Model-Based and Semi-Parametric Estimation of Time Series Components and Mean Square Error of Estimators

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Abstract
Wayne Fuller is known for his outstanding contributions to three main areas in statistics: Sample surveys, Time series and Measurements errors. This paper will focus on time series analysis and more specifically, on estimation of seasonally adjusted and trend components and the mean square error (MSE) of the estimators. We shall compare the component estimators obtained by application of the X-11 ARIMA method with estimators obtained by fitting state-space models that account more directly for correlated sampling errors. The component estimators and MSE estimators are obtained under a different definition of the target components, which conforms to an original proposition by Wayne Fuller. By this definition the unknown components are defined to be the X-11 estimates of them in the absence of sampling errors and with sufficiently long series for application of the symmetric filters imbedded in this procedure. We propose new MSE estimators with respect to this definition. The performance of the estimators is assessed by using simulated series that approximate a real series produced by the Bureau of Labor Statistics in the U.S.A.

Key words: Bias correction, Mean-Square error, Seasonal Adjustment, Trend.

1. Preface

We consider X-11 ARIMA estimators of the trend and seasonally adjusted series (SA) and estimators obtained by fitting state-space models. We define the target seasonal and trend components to be the hypothetical X-11 estimates of them in the absence of sampling errors, assuming that the time series under consideration is sufficiently long for application of the symmetric filters imbedded in the original X-11 procedure. The mean square error (MSE) of the X-11 ARIMA and state-space model estimators are defined with respect to this definition. We estimate the MSE by conditioning on the target components, thus accounting for possible conditional bias in estimating them. The results are illustrated by use of simulated series that approximate a real series produced by the Bureau of Labor Statistics (BLS) in the U.S.A.

2. Target Components, Bias and MSE of X-11 ARIMA Estimators

2.1 Target components

We begin with the usual notion that an economic time series, $Y_t, t = 1,2,...$ can be decomposed into a trend or trend-cycle component T_t, a seasonal component S_t, and an irregular component, I_t; $Y_t = T_t + S_t + I_t$. Here we consider for simplicity the additive decomposition but the results can be generalized to the multiplicative decomposition

$Y_t = T_t \times S_t \times I_t$ by applying the log transformation and employing similar considerations as in Pfeffermann *et al*. (1995). Very often, the series Y_t is unobserved and the actual available series consists of sample estimates, y_t, obtained from repeated sample surveys, and we assume that y_t can be expressed as the sum of the true population series, Y_t, and a sampling error, ε_t. More generally, the observed series can be viewed as the sum of a signal, G_t, and an error; e_t; $y_t = G_t + e_t$, where the signal and the error may be defined in two ways:

*GE*1. $G_t = T_t + S_t$, $e_t = I_t + \varepsilon_t$; in this case e_t is the combined error of the time series irregular and the sampling error (Pfeffermann, 1994);

*GE*2. $G_t = T_t + S_t + I_t$, $e_t = \varepsilon_t$; in this case the irregular term is part of the signal and e_t is the sampling error (Bell and Kramer 1999).

We assume without loss of generality that the series started at time $-\infty < t_{start} < 1$ but y_t is only observed for the time points $t = 1,...,N$, such that

$$y_t = G_t + e_t, \quad t = \underbrace{t_{start},...,0,}_{unobserved} \underbrace{1,...,N}_{y_t-observed} \underbrace{,N+1,...,\infty}_{unobserved}. \tag{1}$$

Here we assume that e_t is independent of $\mathbf{G} = \{G_t, t = t_{start},...,\infty\}$ for all t with $E(e_t) = 0$, $Var(e_t) < \infty$ although in practice, the sampling error, and in particular the variance of the sampling error sometimes depends on the magnitude of the signal.

In order to estimate the unknown components, the X-11 ARIMA program applies a sequence of moving averages (linear filters) to the observed series. Thus, the X-11 ARIMA estimators of the trend and the seasonal components are approximately,

$$\hat{T}_t = \sum_{k=-(t-1)}^{N-t} w_{kt}^T y_{t+k}, \; \hat{S}_t = \sum_{k=-(t-1)}^{N-t} w_{kt}^S y_{t+k}, \tag{2}$$

where the filter coefficients $\{w_{kt}^T\}$, $\{w_{kt}^S\}$ are defined in general by the program options for the given time interval $t = 1,...,N$, by the ARIMA model used to forecast and backcast the series and by the number of backcasts and forecasts. However, at the central part of the series the filters in (2) are time-invariant and symmetric; $w_{kt}^T = w_k^T$, $w_{-k}^T = w_k^T$ for $a_T \le t \le N - a_T$; $w_{kt}^S = w_k^S$, $w_{-k}^S = w_k^S$ for $a_S \le t \le N - a_S$, where a_T, a_S are defined by the X-11 program options. The length of the symmetric filters is $2a_T + 1$ ($2a_S + 1$) such that $w_{kt}^T = w_k^T = 0$ if $k \notin [-a_T, a_T]$, $w_{kt}^S = w_k^S = 0$ if $k \notin [-a_S, a_S]$. For example, with the default X-11 option $a_S = 84$, but it may be as low as 70 or as high as 149. Note that in the central part of the series the X-11 and X-11 ARIMA estimators are the same, such that the symmetric filters depend only on the X-11 program options and not on the ARIMA extrapolations.

Remark 1. The use of X-11 ARIMA involves also 'non-linear' operations such as the identification and estimation of ARIMA models used for the forecasting and backcasting of the original series, and the identification and gradual replacement of extreme observations. We assume that the time series under consideration is already corrected for extreme values. The effects of the identification and non-linear estimation of ARIMA models are generally minor, see Pfeffermann *et al*. (1995) and Pfeffermann *et al*. (2000).

Definition 1. Assuming $t_{start} < \min(-a_T, -a_S)$ and following Bell and Kramer (1999), we define the trend component at time t to be $T_t^{X11} = \sum_{k=-a_T}^{a_T} w_k^T G_{t+k}$. Analogously, the seasonal component is defined as $S_t^{X11} = \sum_{k=-a_S}^{a_S} w_k^S G_{t+k}$. The target components T_t^{X11} and S_t^{X11} are thus the hypothetical estimates that would be obtained when applying the X-11 symmetric filters to the signal $\mathbf{G}$ at time point t , $t = 1,...,N$. It follows therefore that the observed series may be decomposed as the sum of the 'X-11-trend', T_t^{X11}, the 'X-11-seasonal component', S_t^{X11}, and the 'X-11 error', $e_t^{X11} = y_t - T_t^{X11} - S_t^{X11}$;

$$y_t = T_t^{X11} + S_t^{X11} + e_t^{X11} . \tag{3}$$

Result 1. For $a_T < t \le N - a_T$, $T_t^{X11} = E[\hat{T}_t \mid \mathbf{G}]$ and for $a_S < t \le N - a_S$, $S_t^{X11} = E[\hat{S}_t \mid \mathbf{G}]$, where $\hat{T}_t, \hat{S}_t$ are the X-11 estimators defined in (2) and the expectation is taken over the distribution of the errors $\{e_t, t = 1,...,N\}$. It follows therefore that in the central part of the series, $\max(a_T, a_S) < t \le N - \max(a_T, a_S)$, the X-11 ARIMA estimators of the trend and the seasonal components are almost unbiased with respect to the decomposition (3). They are not exactly unbiased because of the 'nonlinear' operations applied by the procedure, mentioned above.

Remark 2. The decomposition defined by (3) refines a saying by Wayne Fuller many years ago in a private conversation, stating that the seasonal and trend components can be defined by the X-11 output. The refinement is in three aspects: First, we remove the sampling error from the definition (computation) and only consider the signal. Second, noting that the filters at the non-central sections of the series are asymmetric and depend on the time points with data, we define the trend and seasonal components to be the (hypothetical) outputs that would be obtained when applying the symmetric filters to the signal. These hypothetical outputs do not depend on the length of the observed series and for a given signal they are fixed parameters. As mentioned before, the decomposition (3) has been used by Bell and Kramer (1999) with the error defined solely by the sampling error, such that the time series irregular, I_t , is part of the signal; $G_t = T_t + S_t + I_t$ (Definition GE2 of the signal). See Remark 3 below for details of their approach.

2.2 Bias and MSE of X-11 ARIMA estimators

The bias, variance and MSE of the X-11 ARIMA estimators with respect to the decomposition (3), conditional on the signal, are as follows:

$$Bias(\hat{T}_t \mid \mathbf{G}) = E[(\hat{T}_t - T_t^{X11}) \mid \mathbf{G}] = \sum_{k=-(t-1)}^{N-t} w_{kt}^T G_{t+k} - \sum_{k=-a_T}^{a_T} w_k^T G_{t+k} . \tag{4}$$

$$Var[\hat{T}_t \mid \mathbf{G}] = E\{[\sum_{k=-(t-1)}^{N-t} w_{kt}^T y_{t+k} - E(\sum_{k=-(t-1)}^{N-t} w_{kt}^T y_{t+k} \mid \mathbf{G})]^2 \mid \mathbf{G}\}$$

$$= E[\sum_{k=-(t-1)}^{N-t} w_{kt}^{T}(y_{t+k} - G_{t+k})]^2 = E(\sum_{k=-(t-1)}^{N-t} w_{kt}^{T} e_{t+k})^2 \,. \tag{5}$$

$$MSE(\hat{T}_t \mid \mathbf{G}) = E[(\hat{T}_t - T_t^{X11})^2 \mid \mathbf{G}] = Var(\hat{T}_t \mid \mathbf{G}) + Bias^2(\hat{T}_t \mid \mathbf{G}) \,. \tag{6}$$

Similar expressions hold for the seasonal and seasonally adjusted estimators.

The expressions in (4)-(6) are general and apply to any linear filter with arbitrary coefficients $\{w_{kt}^{T}\}$ as defined by the X-11 ARIMA options, the ARIMA model used for extrapolations and the number of forecasts and backcasts. The same holds for other component estimators such as the seasonally adjusted series. In fact, as emphasized in Section 3, the expressions in (4)-(6) hold equally for other linear filters, not necessarily imbedded in the X-11 ARIMA program. In the next sections we discuss ways of estimating the MSE in (6).

Remark 3. As noted before, Bell and Kramer (1999) use a similar definition of the target components. The authors estimate these components by augmenting the series with $m = max(a_T, a_S)$ minimum mean squared error forecasts and backcasts under an appropriate model, such that the symmetric filters can be applied to the augmented series at every time point t with observation. The trend estimator, for example, can be written then as $\hat{T}_t^{BK} = \sum_{k=-a_T}^{a_T} w_k^T y_{t+k}^{*}$, where $y_{t+k}^{*} = y_{t+k}$ if y_{t+k} is observed ($1 \le t+k \le N$), and y_{t+k}^{*} is the forecasted or backcasted value otherwise. The authors focus on the variance $Var(\hat{T}_t^{BK} - T_t^{X11})$ under the GE2 definition of the signal as the measure of error, with the variance taken over the distributions of the sampling errors and the forecast and backcast prediction errors. Notice that for unbiased predictions,

$$E(\hat{T}^{BK} - T_t^{X11}) = E[\sum_{k=-a_T}^{a_T} w_k^T y_{t+k}^{*} - \sum_{k=-a_T}^{a_T} w_k^T G_{t+k}] = E[\sum_{k=-a_T}^{a_T} w_k^T y_{t+k}^{*} - \sum_{k=-a_T}^{a_T} w_k^T y_{t+k}] = 0 \,,$$

such that the estimators of the trend are unbiased unconditionally. However, when conditioning on the signal $\mathbf{G} = \{G_t, t = t_{start}, ..., \infty\}$, in general $E[(\hat{T}_t^{BK} - T_t^{X11}) \mid \mathbf{G}] \ne 0$. As is evident from (4), a bias may exist also when forecasting and backcasting less than m observations, even unconditionally, depending on the distribution of the signal.

Our approach differs therefore from Bell and Kramer (1999) in three main aspects. First and for most, our definition of the MSE and its estimation (see below) is not restricted to the case of full forecasts and backcasts, and can be applied for any linear estimator of the form $\tilde{H}_t = \sum_{k=-(t-1)}^{N-t} h_{kt} y_{t+k}$. In particular, it applies to the case when estimating the seasonally adjusted and trend series by use of X-11 ARIMA with only one or two years of forecasts and backcasts (to the best of our knowledge, the common case in practice) or even without ARIMA extrapolations, or when estimating the components by fitting a state-space model to the series, see Section 3. Second, we attempt to estimate the conditional MSE given the signal, even though the signal is in fact unobserved. We believe that many users of seasonally adjusted and trend estimators would feel most comfortable with the notion that the corresponding target components are fixed parameters, which conforms with classical sampling theory, although as stated in Remark 5 below, our bias estimators may also be viewed as estimating the unconditional bias over all possible realizations of the signal under an appropriate model given the observed series. Third, our approach is applicable also to the case where the signal consists of only the trend and the seasonal effect, and the time series irregular is part of the error (GE1 definition of the signal and error).

2.3 Variance estimation

Under the GE2 definition of the signal and error, $e_t = \varepsilon_t$, the sampling error, and by (5), $Var(\hat{T}_t \mid \mathbf{G}) = E(\sum_{k=-(t-1)}^{N-t} w_{kt}^T \varepsilon_{t+k})^2 = \sum_k \sum_l w_{kt}^T w_{lt}^T Cov(\varepsilon_{t+k}, \varepsilon_{t+l})$. Similar expressions apply when estimating the seasonal or the seasonally adjusted value. We assume the availability of estimates of the variances and covariances of the sampling errors for estimating the variance of the estimators.

Next consider the estimation of the variance under the GE1 definition of the signal and error by which $G_t = S_t + T_t$ and $e_t = I_t + \varepsilon_t$. By (5), the variance of the X-11 ARIMA estimator of the trend is a linear combination of the covariances $v_{tm} = Cov(e_t, e_m)$, $t, m = 1,..., N$. Following Pfeffermann (1994), let $R_t = y_t - \hat{S}_t - \hat{T}_t = \sum_{k=-(t-1)}^{N-t} w_{kt}^R y_{t+k}$ define the linear approximation of the X-11 ARIMA residual term at time t, where $w_{0t}^R = 1 - w_{0t}^S - w_{0t}^T$, $w_{kt}^R = -w_{kt}^S - w_{kt}^T$, $k \neq 0$. Then,

$$\begin{aligned} Var(R_t \mid \mathbf{G}) &= E\{[\sum_{k=-(t-1)}^{N-t} w_{kt}^R (y_{t+k} - E(y_{t+k} \mid \mathbf{G})]^2 \mid \mathbf{G}\} = Var \sum_{k=-(t-1)}^{N-t} w_{kt}^R e_{t+k} \\ Cov(R_t, R_m \mid \mathbf{G}) &= Cov(\sum_{k=-(t-1)}^{N-t} w_{kt}^R e_{t+k}, \sum_{l=-(m-1)}^{N-m} w_{lm}^R e_{m+l}) = \sum_k \sum_l w_{kt}^R w_{lm}^R Cov(e_{t+k}, e_{m+l}) \end{aligned} \quad . \qquad (7)$$

By (7), the vector $\mathbf{U}$ of the covariances $u_{tm} = Cov(R_t, R_m \mid \mathbf{G})$ and the vector $\mathbf{V}$ of the covariances $v_{tm} = Cov(e_t, e_m)$, $t, m = 1,..., N$, are related by a system of linear equations,

$$\mathbf{U} = D\mathbf{V}, \qquad (8)$$

where the matrix D is defined by the known weights $\{w_{kt}^R\}$, see Pfeffermann (1994) and Pfeffermann and Scott (1997) for details. Since the X-11 ARIMA residuals are known for $t = 1,..., N$, the covariances $Cov(R_t, R_m \mid \mathbf{G})$ can be estimated from these residuals. Substituting the estimates in the vector $\mathbf{U}$ in (8) allows in theory to estimate $\mathbf{V}$ by solving the resulting equations (D is known). Notice that the use of (8) does not require the availability of estimates of the variances and covariances of the sampling errors. However, the X-11 ARIMA residual series is stationary only in the center of the series and the estimators obtained this way are very unstable. A possible solution to this problem is to assume that the covariances v_{tm} are negligible if $|t-m| > C$ for some constant C and hence can be set to zero, and then solve the reduced set of equations obtained from (8). Additionally, when estimates for the autocovariances of the sampling errors are available, they can be substituted into the vector $\mathbf{V}$, in which case one only needs to estimate the unknown variance and covariances of the time series irregular terms, which reduces the number of unknown covariances and hence the number of equations very drastically. Note that all these procedures are basically ‘model free’. See Pfeffermann (1994), Pfeffermann and Scott (1997) and Chen *et al*. (2003) for different approaches of estimating $\mathbf{U}$ and $\mathbf{V}$. Bell and Kramer (1999) consider model based estimation of the variance and covariances of the sampling errors.

2.4 Bias and MSE estimation

Estimation of the conditional bias given the signal and hence the conditional MSE of the estimator $\hat{S}_t$ (or any other linear estimator) is more involved. In our simulation study we estimate the bias by estimating the signal and then substituting the estimate in the right hand side of the bias expression (4). Within the X-11 ARIMA framework the signal can be estimated most conveniently by application of the following two steps.

(a) Fit a model to the original series and use the X-11 ARIMA forecast–backcast option to augment the series with $m = max(a_T, a_S)$ forecasts and backcasts, thus allowing to apply the symmetric filters for estimating the signal in the central N time points. The length of the augmented series is $N^{aug} = N + 2m$.

(b) Estimate the signal of the augmented series as,

$$\hat{G}_t^{aug} = \sum_{k=-(t-1}^{N^{aug}-t} w_{kt}^{S,aug} y_{t+k}^{aug} + \sum_{k=-(t-1)}^{N^{aug}-t} w_{kt}^{T,aug} y_{t+k}^{aug}, \quad t = -m+1,...,N+m, \tag{9}$$

where $y_t^{aug} = y_t$ if y_t is observed, and y_t^{aug} is the corresponding forecasted (backcasted) value otherwise, and $w_{kt}^{S,aug}, w_{kt}^{T,aug}$ are the X-11 weights corresponding to the augmented series.

Substituting the augmented signal into (4) yields the bias estimate,

$$Bi\hat{a}s[\hat{T}_t \mid \mathbf{G}] = \hat{E}[(\hat{T}_t - T_t^{X11}) \mid \mathbf{G}] = \sum_{k=-(t-1)}^{N-t} w_{kt}^T \hat{G}_{t+k}^{aug} - \sum_{k=-a_T}^{a_T} w_k^T \hat{G}_{t+k}^{aug}, \quad t = 1,...,N. \tag{10}$$

Alternatively, the signal can be estimated more efficiently by extracting the models for the trend and the seasonal effects using signal extraction methodology and then estimate the signal within the observation period, and forecast and backcast the signal under the extracted model. Software for signal extraction is now available within X-13 ARIMA-SEATS **(**X-13A-S Reference Manual). The estimated signal is in this case the minimum mean squared estimate under the model. One may also estimate the signal outside the X-11 ARIMA framework using a different class of models.

Having estimated the conditional bias generally produces a conservative estimator for the conditional MSE defined by (6);

$$M\hat{S}E(\hat{T}_t \mid \mathbf{G}) = V\hat{a}r(\hat{T}_t \mid \mathbf{G}) + Bi\hat{a}s^2(\hat{T}_t \mid \mathbf{G}). \tag{11}$$

The estimator in (11) is conservative since $E[Bi\hat{a}s^2(\hat{T}_t \mid \mathbf{G}) \mid \mathbf{G}] = \{E[Bi\hat{a}s(\hat{T}_t \mid \boldsymbol{G})] \mid \boldsymbol{G}\}^2$ $+Var[Bi\hat{a}s(\hat{T}_t \mid \mathbf{G}) \mid \mathbf{G}] \geq \{E[Bi\hat{a}s(\hat{T}_t \mid \boldsymbol{G})] \mid \boldsymbol{G}\}^2$. The overestimation of the MSE can be corrected by subtracting an estimate of $Var[Bi\hat{a}s(\hat{T}_t \mid \mathbf{G}) \mid \mathbf{G}]$. Notice that $Bi\hat{a}s(\hat{T}_t \mid \mathbf{G})$ is again a linear combination of all the observed values so the variance can be estimated similarly to the estimation of $Var[\hat{T}_t \mid \mathbf{G}]$ discussed in Section 2.3.

Remark 4. A possible objection to estimating the signal many years ahead is that the signal estimators may be severely biased for time points far away from the last time point N with observation. Note, however, that the signal estimates in (9) may be biased (given the true signal) but the bias estimator in (10) may still be unbiased or only have a small bias. For example, if $E(\hat{G}_t^{aug} \mid \mathbf{G}) = B$ for all t where B is a constant, the bias estimator in

(10) is unbiased for the true bias since $\sum_{k=-(t-1)}^{N-t} w_{kt}^{T} = \sum_{k=-a_T}^{a_T} w_{k}^{T} = 1$, and similarly for the seasonal adjustment filter. In addition, the weights of the symmetric and asymmetric filters decay to zero very fast when moving away from the time point of interest, so that even large biases for the estimators of the signal for months far away from the last time point with observation may have little effect on the bias of the bias estimator in (10). See Figure 1 for the weights of the linear filters used in our study.

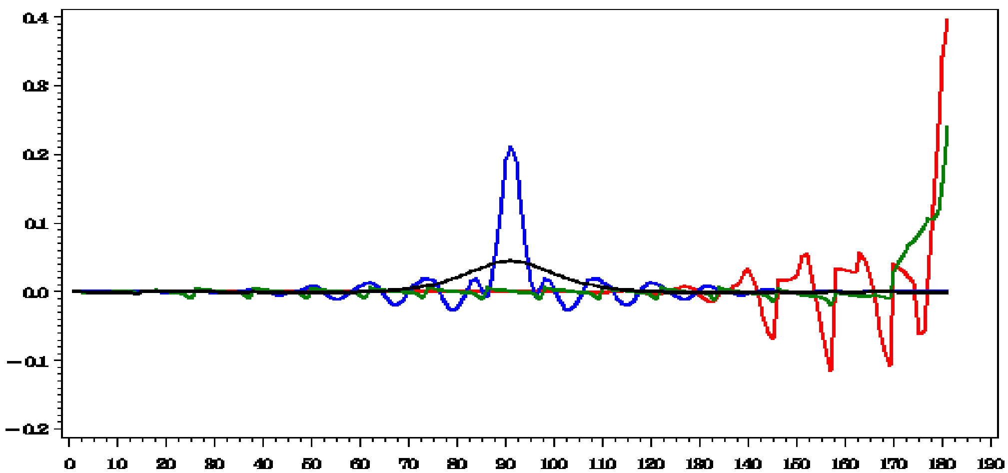

Figure 1. Central and end weights when estimating the trend by default X-11 and under Basic Structural Model.

Remark 5. When the signal is estimated by the minimum mean squared estimate under a model, the estimate coincides with the conditional expectation given the observed series. In this case the bias estimator (10) is the conditional expectation of the bias over all possible realizations of the signal given the observed series, implying an 'unconditional' interpretation for the bias estimator.

3. The use of State-Space Models for Estimation of the X-11 Trend and Seasonally Adjusted Series

Consider any other set of component estimators of the form,

$$\tilde{T}_t = \sum_{k=-(t-1)}^{N-t} h_{kt}^{T} y_{t+k} \ , \quad \tilde{S}_t = \sum_{k=-(t-1)}^{N-t} h_{tk}^{S} y_{t+k} \ . \tag{12}$$

Then, similar to the case of X-11 ARIMA estimators considered in Section 2, we can calculate the conditional bias and MSE with respect to the X-11 components defined in Definition 1, that is,

$$Bias(\tilde{T}_t \mid \mathbf{G}) = E[(\tilde{T}_t - T_t^{X11}) \mid \mathbf{G}] = \sum_{k=-(t-1)}^{N-t} h_{kt}^{T} G_{t+k} - \sum_{k=-a_T}^{a_T} w_{k}^{T} G_{t+k} \ . \tag{13}$$

$$Var[\tilde{T}_t \mid \mathbf{G}] = E(\sum_{k=-(t-1)}^{N-t} h_{kt}^{T} e_{t+k})^2 \ . \tag{14}$$

$$MSE(\tilde{T}_t \mid \mathbf{G}) = E[(\tilde{T}_t - T_t^{X11})^2 \mid \mathbf{G}] = Var(\tilde{T}_t \mid \mathbf{G}) + Bias^2(\tilde{T}_t \mid \mathbf{G}) \,. \tag{15}$$

In particular, when fitting a state-space model with given (estimated) hyper-parameters, the state-space model estimates of the seasonal effects and the trend for any given time t are again linear combinations of the observed series. In our empirical study we calculated the weights defining the corresponding filters by using the impulse response method (Findley and Martin, 2006). By this method the weight of an observation for time *m* when applying the filter at time *t* is obtained by fitting the model with the observation at time *m* set to 1 and all other observations set to zero, and then computing the estimate for time *t*. Calculation of all the weights for all the time points for a series of length *N* requires therefore fitting the model *N* times, each time with a vector observation defined by a different column of the identity matrix I_N. Substituting the weights in (13)-(15) defines the corresponding bias and MSE. As mentioned before, the bias can be estimated in this case by estimating the signal $\mathbf{G}^{aug} = (G_{-a_S+1},...,G_0,...,G_N,...,G_{N+a_S})$ optimally under the model. See next section for details of the model used in our empirical application. The bias and MSE estimators are obtained similarly to Equations (10) and (11).

4. Simulation Study

In this section we apply the estimators considered in Sections 2 and 3 to simulated series, generated from a model fitted by the Bureau of Labour Statistics (BLS) in the U.S.A. to the series *Employment to Population Ratio in the District of Columbia*, abbreviated hereafter by EP-DC. The EP series represents the percentage of employed persons out of the total population aged 15+. This is one of the key economic series in the U.S.A., produced monthly by the BLS for each of the 50 States and DC. The BLS uses similar models for the production of the major employment and unemployment estimates in all the States, see Tiller (1992) for details. In order to assess the performance of the various estimators, we generated a large number of series from the EP-DC model. The model contains 18 unknown hyper-parameters estimated in 3 stages, but for the present experiment we consider the hyper-parameter estimates as the true parameters. See Pfeffermann and Tiller (2005) for the parameter estimation procedures used by the BLS.

4.1 Model fitted

The EP-DC series is plotted in Figure 2 along with the estimated trend under the state-space model defined below and the trend estimated by application of X-11 ARIMA with 12 months forecasts. In this presentation we only consider the estimation of the trend in the last four years of data so no backcasts were needed. This is a very erratic series: the residual component (calculated by X11 ARIMA) explains 55% of the month to month changes and 32% of the yearly changes. A large portion of the residual component is explained by sampling errors. Let y_t define the direct sample estimate at time *t* and Y_t the corresponding true population ratio such that $\varepsilon_t = y_t - Y_t$ is the sampling error. A state-space model is fitted to the series y_t that combines a model for Y_t with a model for ε_t. The model postulated for Y_t is the Basic Structural Model (BSM, Harvey, 1989),

$$\begin{aligned} &Y_t = T_t + S_t + I_t \,, \;\; I_t \sim N(0,\sigma_I^2) \\ &T_t = T_{t-1} + R_t \;\;\; ; \;\;\; R_t = R_{t-1} + \eta_{Rt} \;\;\; , \;\;\; \eta_{Rt} \sim N(0,\sigma_R^2) \\ &S_t = \sum\nolimits_{j=1}^{6} S_{j,t} \,; \end{aligned} \tag{16}$$

$$S_{j,t} = \cos\omega_j S_{j,t-1} + \sin\omega_j S^*_{j,t-1} + \eta_{j,t} \ , \ \eta_{j,t} \sim N(0,\sigma_S^2)$$
$$S^*_{j,t} = -\sin\omega_j S_{j,t-1} + \cos\omega_j S^*_{j,t-1} + \eta^*_{j,t} \ , \ \eta^*_{j,t} \sim N(0,\sigma_S^2)$$
$$\omega_j = 2\pi_j / 12 \, , \, j = 1 \ldots 6$$

The error terms $I_t, \eta_{Rt}, \eta_{j,t}, \eta^*_{j,t}$ are mutually independent normal disturbances. In the model (16), T_t is the trend level, R_t is the slope and S_t is the seasonal effect. The model for the trend approximates a local linear trend, whereas the model for the seasonal effects uses the traditional decomposition of the seasonal component into 11 cyclical components corresponding to the 6 seasonal frequencies. The added noise enables the seasonal effects to evolve stochastically over time.

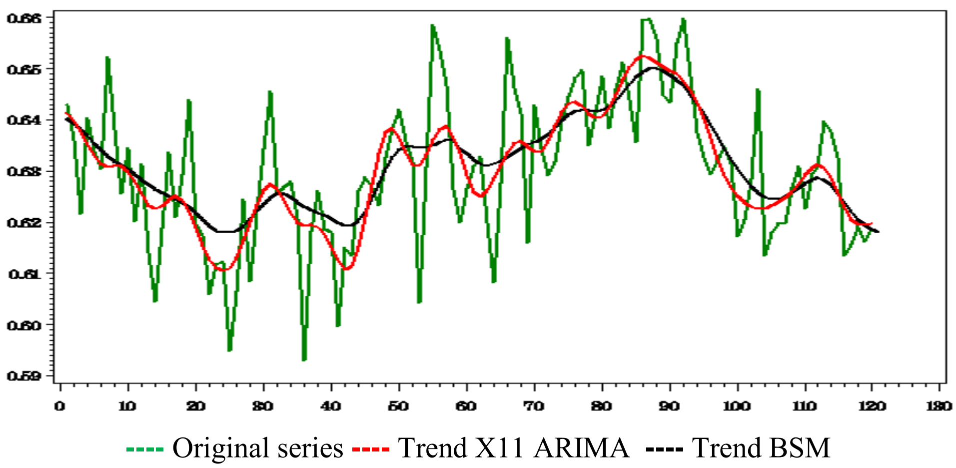

Figure 2. Employment to Population Ratio in DC, 1999-2000. Original series and trends estimated by X-11 ARIMA with 12 forecasts and under BSM.

The model fitted for the sampling error is *AR*(15), which approximates the sum of an *MA*(15) process and an *AR*(2) process. The *MA*(15) process accounts for the autocorrelations implied by the sample overlap resulting from the Labor Force Survey rotating sampling scheme. By this scheme, households in the sample are surveyed for 4 successive months, they are left out of the sample for the next 8 months and then they are surveyed again for 4 more months. The *AR*(2) process accounts for the autocorrelations arising from the fact that households dropped from the survey are replaced by households from the same 'census tract'. These autocorrelations exist irrespective of the sample overlap. The reduced ARMA representation of the sum of the two models is *ARMA*(2,17), which is approximated by an *AR*(15) model.

The separate models holding for the population ratios and the sampling errors are cast into a single state-space model. In what follows we refer to the combined model holding for the observed series as the BSM model. Note that the state vector consists of the trend, slope, seasonal effects and sampling errors. The monthly variances of the sampling errors are estimated externally based on a large number of replications and are considered as known, implying that the combined model depends on 18 hyper-parameters.

Remark 6. In this simulation study we used the known (previously estimated) hyper-parameters and did not re-estimate them, except when predicting the signal values for bias estimation, in which case we re-estimated the BSM hyper-parameters for every simulated series.

4.2 Simulation plan

We generated three sets of series. The *first set* consists of 1,000 series of observations $y_t^b, b=1,...,1,000$, each of length 301, obtained by simulating separately for every month $t=1,...,301$ a trend, seasonal effects and irregular term from the model (16), and a sampling error from the AR(15) model, and then summing the separate components; $y_t^b = T_t^b + S_t^b + I_t^b + \varepsilon_t^b$. For the present study we employ the GE2 definition of the signal such that $G_t = T_t + S_t + I_t$. A *second set* of 1,000 series was obtained by fixing the signal of the series generated in the second replication of the first set and adding the sampling errors from the first set to the fixed signal. Thus the second set of series has the same sampling errors as the first set but all the series have the same signal. The third set of series is similar to the first set, except that the trend was generated deterministically as $T_t = 60 - 10\times[2.8 - \exp(t/100) - \exp[(t/100)^2/3] + \exp\{[(180-t)/100]^3/4\}]$. The other components are the same as for the second set. The use of this trend allows studying the robustness of the estimators to model misspecification.

Next we computed the default X-11 estimator of the trend and the seasonal component for each simulated signal in order to obtain the target X-11 components defined by (3) for the central 181 months. We defined the target for the seasonally adjusted component as the difference between the original series without sampling error and the target for the seasonal, that is,

$$SA_t^{X11} = (y_t - S_t^{X11} - \varepsilon_t);\ \ S_t^{X11} = \sum_{k=-a_S}^{a_S} w_k^S G_{t+k}\ ,\quad T_t^{X11} = \sum_{k=-a_T}^{a_T} w_k^T G_{t+k}\ . \tag{17}$$

Finally, we removed the first and last 60 monthly observations from the simulated series and applied X-11 ARIMA with only 12 forecasts and with 60 forecasts using the default X-11 filters, but fixing the ARIMA model to be the airline model (0,1,1),(0,1,1). The programme estimated the model parameters for each series. Here we only show and discuss the results for the last 6 months of data (last 4 years in Figure 3) and so no backcasts were needed. Additionally, we fitted the BSM for each of the reduced series of length $N=181$. The resulting estimators are,

$$\begin{aligned} \hat{T}_{t,b}^{X11} &= \sum_{k=-(t-1)}^{N-t} w_{kt}^T y_{t+k}^b, & \hat{SA}_{t,b}^{X11} &= y_t^b - \sum_{k=-(t-1)}^{N-t} w_{kt}^S y_{t+k}^b; \\ \hat{T}_{t,b}^{BSM} &= \sum_{k=-(t-1)}^{N-t} h_{kt}^T y_{t+k}^b, & \hat{SA}_t^{BSM} &= y_t^b - \sum_{k=-(t-1)}^{N-t} h_{kt}^{SA} y_{t+k}^b; \end{aligned}\ , \tag{18}$$

where the weights for the X-11 ARIMA estimates are defined by the number of forecasts (12 or 60) and $\{h_{kt}^S\}$, $\{h_{kt}^T\}$ are the weights defining the estimators of the seasonal component and the trend under the BSM.

4.3 Computations

We computed the empirical bias and MSEs for the first second and third sets of 1,000 series. The first set of empirical bias and MSE correspond to the unconditional bias and MSE, integrated over the distribution of the signal, where as the second and third sets of bias and MSE condition on a given signal. In what follows we define the computations performed.

Step 1. Compute the variance of X-11 ARIMA and BSM estimators:

$$V_t^{T,X11} = \sum_k \sum_l w_{kt}^T w_{lt}^T Cov(\varepsilon_{t+k}, \varepsilon_{t+l}), \quad V_t^{SA,X11} = \sum_k \sum_l w_{kt}^{SA} w_{lt}^{SA} Cov(\varepsilon_{t+k}, \varepsilon_{t+l}),$$
$$V_t^{T,BSM} = \sum_k \sum_l h_{kt}^T h_{lt}^T Cov(\varepsilon_{t+k}, \varepsilon_{t+l}), \quad V_t^{SA,BSM} = \sum_k \sum_l h_{kt}^{SA} h_{lt}^{SA} Cov(\varepsilon_{t+k}, \varepsilon_{t+l}).$$

Notice that the variances do not depend on the actual observed (simulated) series.

Step 2. Compute the empirical bias and root mean square error (RMSE) over the 1,000 replications:

Empirical Bias of X-11 ARIMA estimators with respect to proposed target (*Equation* 3),

$$Bias(\hat{T}_t^{X11}) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} (\hat{T}_{t,b}^{X11} - T_{t,b}^{X11}), \ Bias(S\hat{A}_t^{X11}) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} (S\hat{A}_{t,b}^{X11} - SA_{t,b}^{X11}), \tag{19}$$

where $\hat{T}_{t,b}^{X11}$ and $S\hat{A}_{t,b}^{X11}$ are defined by (18) and the target components are defined by (17).

Empirical RMSE of X-11 ARIMA estimators with respect to proposed target,

$$\begin{aligned} RMSE(\hat{T}_t^{X11}) &= \sqrt{\frac{1}{1{,}000} \sum_{b=1}^{1{,}000} (\hat{T}_{t,b}^{X11} - T_{t,b}^{X11})^2} \\ RMSE(S\hat{A}_t^{X11}) &= \sqrt{\frac{1}{1{,}000} \sum_{b=1}^{1{,}000} (S\hat{A}_{t,b}^{X11} - SA_{t,b}^{X11})^2} \end{aligned}. \tag{20}$$

The empirical bias and RMSE of the BSM estimators are computed the same way.

Step 3. Compute the estimators of the bias and RMSE:
Denote by $\hat{B}_{t,b}^T$ and $\hat{B}_{t,b}^{SA}$ the bias estimates defined by (10) with the signal estimated by using X-11 ARIMA. The MSE estimates when applying the ARIMA procedure are computed as $M\hat{S}E_{t,b}^{T,X11} = V_t^{T,X11} + (\hat{B}_{t,b}^{T,X11})^2$, $M\hat{S}E_{t,b}^{SA,X11} = V_t^{SA,X11} + (\hat{B}_{t,b}^{SA,X11})^2$. The MSE estimates when fitting the state-space model are computed in the same way. Finally, compute:

Average of the Bias estimates,

$$Av(\hat{B}_{t,b}^T) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} \hat{B}_{t,b}^T, \quad Av(\hat{B}_{t,b}^{SA}) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} \hat{B}_{t,b}^{SA}; \tag{21}$$

Empirical Standard Error of the Average of the Bias estimates,

$$\begin{aligned} SE(\hat{B}_{t,b}^T) &= \sqrt{\frac{1}{1{,}000^2} \sum_{b=1}^{1{,}000} (\hat{B}_{t,b}^T - \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} \hat{B}_{t,b}^T)^2} \\ SE(\hat{B}_{t,b}^{SA}) &= \sqrt{\frac{1}{1{,}000^2} \sum_{b=1}^{1{,}000} (\hat{B}_{t,b}^{SA} - \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} \hat{B}_{t,b}^{SA})^2} \end{aligned}; \tag{22}$$

Average of MSE estimates,

$$Av(M\hat{S}E_{t,b}^T) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} M\hat{S}E_{t,b}^T, \quad Av(M\hat{S}E_{t,b}^{SA}) = \frac{1}{1{,}000} \sum_{b=1}^{1{,}000} M\hat{S}E_{t,b}^{SA}. \tag{23}$$

Remark 7. The statistics computed for the first set of 1,000 series represent the unconditional bias and MSE. For the second and third data sets we repeated the same computations but the empirical bias and MSE are now conditional on the given signal.

4.4 Results

The results are summarized in Tables 1 – 3 and Figure 3. We only show the results pertaining to trend estimation. All the statistics in the tables below are multiplied by 100.

Table 1: Empirical bias and RMSE, and mean estimates as obtained by application of X-11 ARIMA and by fitting the BSM. First set of series, last six months.

		Jul	Aug	Sep	Oct	Nov	Dec
X-11 ARIMA 60 forecasts	Empirical Bias, Eq. 19	0.02	0.03	0.03	0.03	0.03	0.02
	Average of Bias Estimates, Eq.21 (SE, Eq. 22)	-0.01 (0.01)	-0.01 (0.01)	-0.01 (0.01)	-0.01 (0.01)	0 (0.01)	0 (0.01)
	Empirical RMSE, Eq. 20	1.26	1.27	1.27	1.28	1.35	1.47
	Average of RMSE Estimates, Eq.23	1.27	1.27	1.27	1.28	1.32	1.37
X-11 ARIMA 12 forecasts	Empirical Bias	0.02	0.03	0.04	0.04	0.03	0.02
	Average of Bias Estimates (SE)	0 (0.01)	0 (0.01)	0 (0.01)	0 (0.01)	0 (0.01)	0.01 (0.01)
	Empirical RMSE	1.27	1.27	1.27	1.29	1.35	1.48
	Average of RMSE Estimates	1.26	1.27	1.27	1.28	1.31	1.37
BSM	Empirical Bias	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01
	Average of Bias Estimates (SE)	0.01 (0.02)	0 (0.01)	0 (0.01)	0 (0.01)	0 (0.02)	0.01 (0.02)
	Empirical RMSE	1.11	1.14	1.16	1.21	1.29	1.39
	Average of RMSE Estimates	1.10	1.12	1.15	1.19	1.29	1.47

All the estimates of the bias in Table 1 are very close to the corresponding empirical bias, indicating no bias in the bias estimation. Recall that the results in Table 1 represent the unconditional bias and MSE. In this case if the ARIMA model used for the forecasts and the BSM are correct, then the X-11 ARIMA and BSM estimators of the trend are unbiased unconditionally.

Table 2 corresponds to the case of a fixed signal. In this particular simulation the absolute empirical biases of the BSM estimators are in most cases larger than the biases of X-11 ARIMA estimators with 12 forecasts, and the latter estimators have larger biases than the X-11 ARIMA estimators with 60 forecasts. In fact, the X-11 ARIMA estimators with 60 forecasts only have a significant bias for the last month whereas the BSM estimators are biased for all 6 months. The empirical statistics and the averages of their estimators look similar in most cases. It is interesting to note that for most of the months the RMSE's of the BSM estimators are lower than RMSE's of X-11 ARIMA estimators, even though the latter estimators are unbiased in the central part of the series.

Table 2: Empirical bias and RMSE, and mean estimates as obtained by application of X-11 ARIMA and by fitting the BSM. Second set of series, last six months.

		Jul	Aug	Sep	Oct	Nov	Dec
X-11 ARIMA 60 forecasts	Empirical Bias, Eq. 19	-0.06	-0.04	-0.02	-0.02	0.05	0.2
	Average of Bias Estimates, Eq.21 (SE, Eq. 22)	-0.02 (0.01)	-0.04 (0.01)	-0.06 (0.01)	-0.03 (0.01)	0.10 (0.01)	0.34 (0.01)
	Empirical RMSE, Eq. 20	1.26	1.26	1.25	1.27	1.30	1.35
	Average of RMSE Estimates, Eq.23	1.26	1.26	1.26	1.27	1.31	1.39
X-11 ARIMA 12 forecasts	Empirical Bias	-0.12	-0.11	-0.07	-0.06	0.03	0.21
	Average of Bias Estimates (SE)	-0.08	-0.08	-0.07	-0.02	0.13	0.38
	Empirical RMSE	1.26	1.27	1.26	1.27	1.30	1.36
	Average of RMSE Estimates	1.26	1.27	1.27	1.27	1.31	1.4
BSM	Empirical Bias	-0.01	-0.22	-0.49	-0.74	-0.74	-0.47
	Average of Bias Estimates (SE)	0 (0.02)	-0.2 (0.01)	-0.38 (0.01)	-0.53 (0.01)	-0.59 (0.02)	-0.56 (0.02)
	Empirical RMSE	1.01	1.07	1.18	1.34	1.38	1.29
	Average of RMSE Estimates	1.01	1.06	1.15	1.24	1.31	1.33

The results in Table 3 again correspond to a conditional case (fixed signal) but now the model is misspecified. This is the reason why the bias estimates perform badly for the last 6 months under all the methods, see Remark 4 above. Note that the BSM estimators are biased with respect to the target even in the middle of the series; see Figure 1 for the central weights used under the BSM and X-11 ARIMA. Figure 3 shows the empirical bias and the average of the BSM bias estimates for the last 4 years of data and it is evident that the bias estimators perform well except in the last 18 months or so.

Table 3: Empirical bias and RMSE, and mean estimates as obtained by application of X-11 ARIMA and by fitting the BSM. Third set of series, last six months.

		Jul	Aug	Sep	Oct	Nov	Dec
X-11 ARIMA 60 forecasts	Empirical Bias, Eq. 19	-0.13	-0.12	-0.07	0.03	0.32	0.83
	Average of Bias Estimates, Eq.21 (SE, Eq. 22)	0.03 (0.01)	0.05 (0.01)	0.05 (0.01)	0.06 (0.01)	0.1 (0.01)	0.17 (0.01)
	Empirical RMSE, Eq. 20	1.27	1.27	1.26	1.27	1.33	1.56
	Average of RMSE Estimates, Eq.23	1.26	1.26	1.26	1.27	1.30	1.33
X-11 ARIMA 12 forecasts	Empirical Bias	-0.21	-0.2	-0.14	-0.01	0.3	0.85
	Average of Bias Estimates (SE)	-0.04 (0.01)	-0.05 (0.01)	-0.04 (0.01)	-0.01 (0.01)	0.07 (0.01)	0.2 (0.01)
	Empirical RMSE	1.28	1.28	1.27	1.27	1.33	1.57
	Average of RMSE Estimates	1.26	1.27	1.26	1.27	1.30	1.34
BSM	Empirical Bias	-0.09	-0.14	-0.21	-0.26	-0.05	0.44
	Average of Bias Estimates (SE)	0.06 (0.02)	0.04 (0.01)	-0.01 (0.01)	-0.07 (0.01)	-0.12 (0.01)	-0.15 (0.02)
	Empirical RMSE	1.01	1.05	1.1	1.15	1.16	1.28
	Average of RMSE Estimates	1.01	1.05	1.08	1.12	1.17	1.22

Intermediate conclusion: The method seems to work well when the model is correctly specified but we need to experiment with many more simulated and real series. The robustness of the method to possible model misspecification needs to be explored further.

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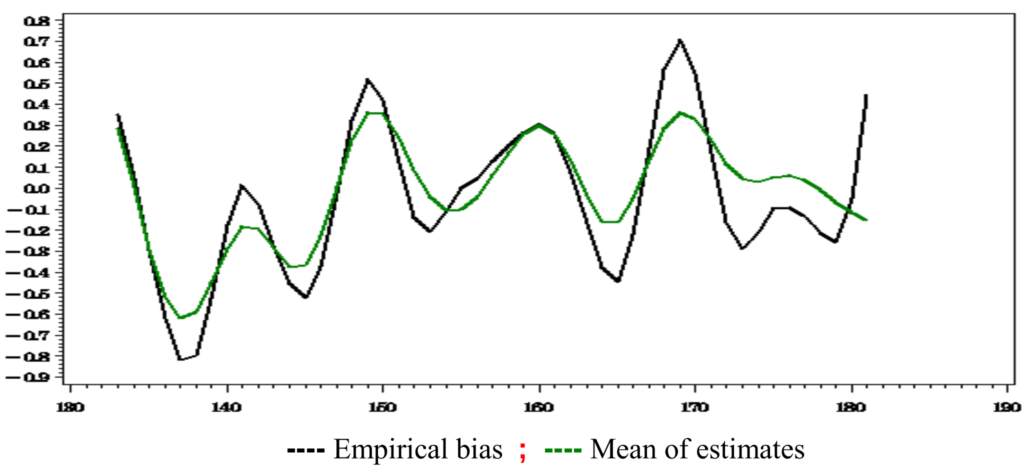

Figure 3. Empirical conditional bias of BSM trend estimates and mean of bias estimates, misspecified trend, signal estimated by X-11 ARIMA.

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