Characteristics of a Model-based Variance Measure for X-11 Seasonal Adjustment October 2010

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1. Introduction

Building from the work of Wolter and Monsour (1981), Pfeffermann (1994) and Bell and Kramer (1999) develop variance measures for X-11 seasonal adjustment which account for both time series features and sampling error. Scott, Pfeffermann, and Sverchkov (2011) summarize methods and a body of empirical work. This paper investigates a puzzling, unsatisfying property that sometimes occurs in time series of variance or standard deviation estimates stemming from the measures. Figure 1 contains graphs of standard deviations for error in seasonal adjustment from two methods for three economic time series. Before explaining these graphs in detail, we focus simply on the shapes. Figure 1a shows values in blue increasing from the center to the ends of the series. In Figure 1b, both methods show a dip during the last year, with one staying below central values throughout the last year and the other increasing to values at the ends exceeding the central values. For the series in Figure 1c, the methods have similar shapes, with dips toward the ends and end values well below the central values. The natural view of seasonal adjustment is that there is more uncertainty at the ends of the series than in the center, that is,

$$SDA(end) > SDA(center) \,. \quad (1)$$

This paper explores when and why deviations from this property occur, as in Figure 1c.

We begin by considering the conventional decomposition of an economic time series Y into a trend or trend-cycle, a seasonal effect, and an irregular or noise term, or, alternatively, into a seasonally adjusted series and a seasonal component,

$$Y = T + S + I = A + S \,.$$

Typically, the population or signal values Y are unknown and the data are obtained from a sample survey, implying that the observed series is $y = Y + \varepsilon$, ε representing the sampling error. Seasonal adjustment is usually carried out by augmenting the series with ARIMA model extrapolations. Let

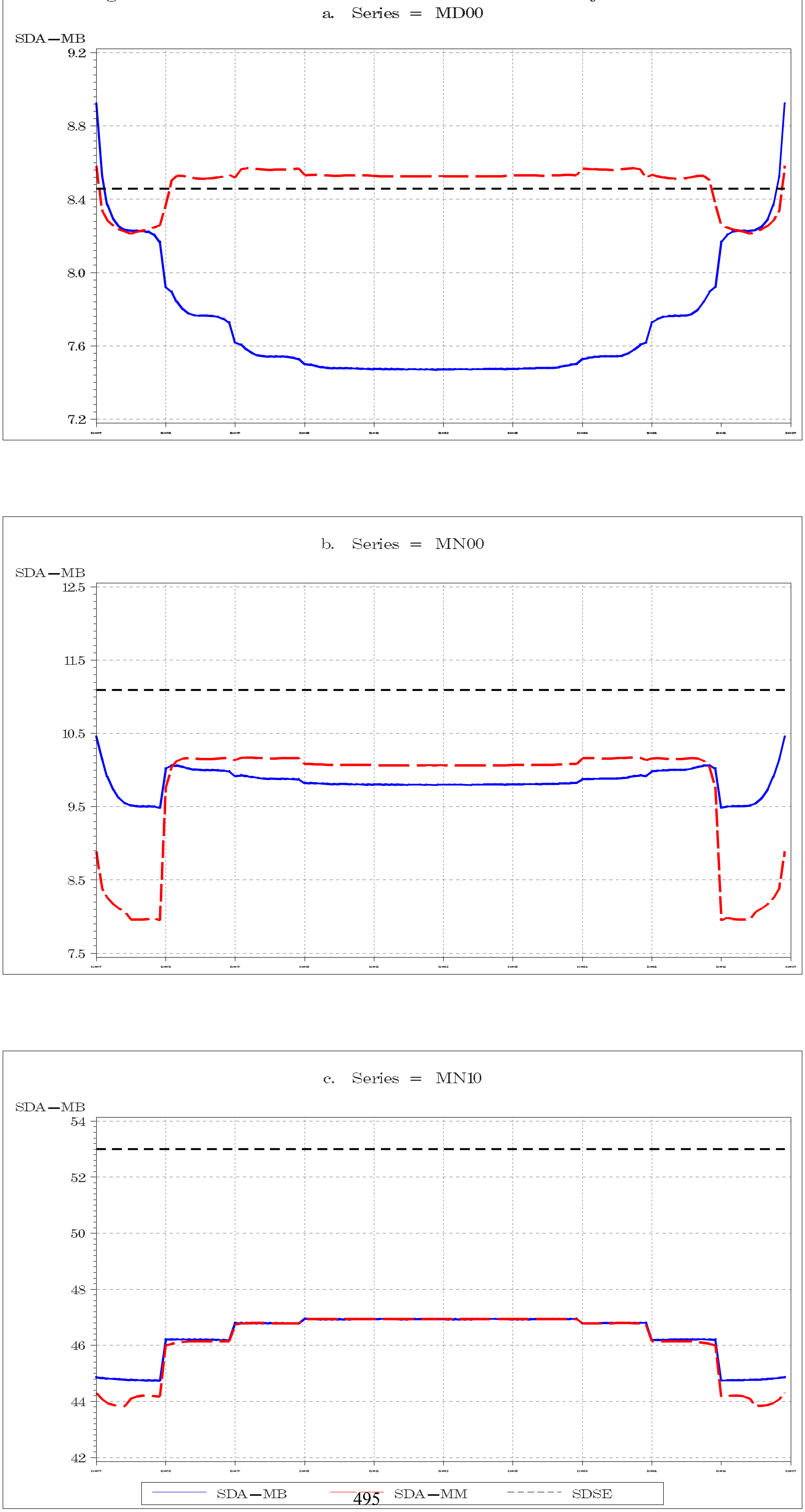
$$\hat{y} = [\hat{y}_{-m+1}, \cdots, \hat{y}_0, y_1, \cdots, y_n, \hat{y}_{n+1}, \cdots, \hat{y}_{n+m}]' = [\hat{y}_b', y_o', \hat{y}_f']'$$

define the $(n+2m)\times 1$ vector consisting of m backcasts, n observed values, and m forecasts. The seasonally adjusted estimator is $\hat{A} = \Omega\,\hat{y}$, where Ω is the $n \times (n+2m)$ matrix of the X-11 seasonal adjustment filter. Bell and Kramer (1999) define the target seasonally adjusted series to be $A^* = \Omega Y$, the seasonally adjusted series that would be obtained if Y were observed for all $n+2m$ time points, and the variance measure to come from the error in estimating this target,

$$Var(A^* - \hat{A}) = Var(\Omega Y - \Omega \hat{y}) = \Omega Var(Y - \hat{y})\Omega' \,.$$

Application of the method requires modeling the signal series Y with an ARIMA model and the sampling error ε with a (stationary) ARMA model. This measure accounts for variability of the sampling errors and for backcast and forecast error. The authors

Figure 1. MB and MM Standard Deviations for Seasonal Adjustment Error



provide all the necessary details for calculating $Var(Y-\hat{y})$. Hereafter we refer to this method as the model-based method (MB).

Pfeffermann (1994) proposes a moments-matching (MM) method for variance estimation and defines the target series to be $\tilde{A} = Y - \tilde{S}$, where $\tilde{S} = \Omega_S(Y-I) = \Omega_S(T+S)$ and Ω_S is the X-11 seasonal filter matrix. Pfeffermann (1994) assumes that the irregular component and the sampling error are stationary and mutually independent, implying that for the combined error $e_t = \varepsilon_t + I_t$,

$$\gamma_k = Cov(e_t, e_{t-k}) = Cov(\varepsilon_t, \varepsilon_{t-k}) + Cov(I_t, I_{t-k}) \equiv \lambda_k + \nu_k,\ k = 0,1,\dots .$$

The following approximation is developed in the paper:

$$Var(\tilde{A}_t - \hat{A}_t) \approx Var(\sum\nolimits_{k=1}^{n} w_{tk}^{(S)} e_k) + \lambda_0(1 - 2w_{tt}^{(S)}) - 2\sum\nolimits_{k \neq t} w_{tk}^{(S)} \lambda_{t-k},$$

where the coefficients $\{w_{tk}^{(S)}\}$ are the weights of the X-11 seasonal filter for time t. This measure accounts for the variability of the sampling error and the irregular component. Assuming the availability of the variance and autocovariances of the sampling errors, estimation of the variance measure requires estimating the vector γ of autocovariances γ_k. Pfeffermann develops these estimates based on a system of linear equations involving empirical moments of the estimated irregular from application of the X-11 method.

Each graph in Figure 1 contains SDA_{MB}, SDA_{MM}, and $SDSE$, square roots of the respective variance measures and the sampling error (SE) standard deviation, assumed constant. In Figure 1a, SDA_{MM} is larger than SDA_{MB} in the center of the series, which illustrates the extra error term which the MM method includes. In all three graphs, we see $SDA_{MB} > SDA_{MM}$ at the ends of the series, showing that the MB method captures more fully forecast-backcast error. As already mentioned, for the series in Figure 1c, both measures fall below the central values at the end.

Section 2 presents empirical results for employment series from the U.S. Bureau of Labor Statistics (BLS), which help explain the behavior seen in Figure 1. Section 3 goes into some details of calculations for the model-based method. Section 4 characterizes end-behavior for a typical example and a final section summarizes results.

2. Empirical End-behavior of Variance Measures for Employment Series

The results in Figure 1 come from three of about 145 industry employment series appearing in BLS's monthly *Employment Situation* press release. A large-scale empirical test of the MB and MM methods has been carried out on monthly change for these series. The data come from BLS's Current Employment Statistics (CES) program, a monthly survey of over 300,000 establishments. In addition to its large size, the survey has the advantage of annual population figures from an external source, the Unemployment Insurance (UI) program. Employers report monthly employment on quarterly tax forms. With a 10-month lag, survey estimates are benchmarked to population UI values. If t is the current month, the employment estimate y_t comes from a "link-relative" estimator,

$$y_t = Y_0 \cdot r_1 \cdot r_2 \cdots r_t,$$

where Y_0 is the latest available benchmark, subsequent subscripts denote number of months away from the benchmark, and

$$r_j = \sum_{i \in M_j} w_{ij} y_{ij} \Big/ \sum_{i \in M_j} w_{i,j-1} y_{i,j-1}$$

is the ratio of weighted employment in months j and $j-1$, with y_{ij} representing the number of employees in establishment i in month j and M_j the set of establishments reporting in both months j and $j-1$. Most national employment series are seasonally adjusted multiplicatively, which fits with modeling the series on the log scale. Monthly change takes the simple form

$$\log(y_t) - \log(y_{t-1}) = \log(y_t / y_{t-1}) = \log(r_t).$$

Sampling error standard deviations and auto correlations are computed each month using the BRR method for various statistics, including these log ratios. Figure 2 is a scatterplot of a set of absolute log ratios (multiplied by 10^4) and the sampling error standard deviations for the Durable Manufacturing series (MD00). The two largest standard deviations occur for large log ratios, but, overall, there is very little pattern. This holds for most of the series examined, which has led us to assume a constant variance for the sampling error. Lag 1 SE autocorrelation estimates vary considerably but tend to be negative. Autocorrelations at other lags are also variable, with medians close to 0. This leads us to adopt an MA(1) model for all series. We compute median lag 1 autocorrelation and round to the nearest .05, except for rounding all magnitudes below .075 to 0 and allowing a maximum magnitude of .20.

Table 1b shows that property (1) holds for the MB method for Manufacturing, Durable Goods (MD00) and for Manufacturing, Nondurable Goods (MN00), but not for Petroleum & Coal Products (MN10), in agreement with Figure 1. For the MM method, (1) holds only for MD00 and then only by 0.7%. Table 1a shows modeling information used for seasonal adjustment and calculation of the two variance measures. For MD00, estimates of the disturbance variance for both the signal model and the irregular component exceed the SE disturbance variance. Sampling error is more prominent for MN00, with the irregular component estimated to be quite small, and dominant for MN10, for which no irregular is identified. It appears that (1) is influenced by the amount of sampling error.

To obtain a measure of SE size, it is fruitful to difference the observed series. Let $\delta(B)$ be the differencing operator in the ARIMA model for the signal. Then,

$$w_t = \delta(B) y_t = u_t + v_t, \quad (2)$$

where $u_t = \psi(B) a_t$, $v_t = \delta(B)\varepsilon_t$, and $\psi(B)$ represents the ARMA model for the differenced signal. An important point in the derivations of Bell and Kramer (1999) is that error in forecasting the series is a function of error in forecasting the differenced series,

$$y_f - \hat{y}_f = C(w_f - \hat{w}_f),$$

for some matrix C. The differenced series is stationary and we can justifiably consider variances. A natural measure of the SE contribution is the "differenced variance ratio,"

$$DVR = Var\left[\delta(B)\varepsilon_t\right] \Big/ Var(w_t). \quad (3)$$

Table 1. Results for MD00, MN00, and MN10

a. Modeling Information

Series	Sampling Error		Irregular	Signal	
	$\hat{\rho}_1$	$\hat{\sigma}^2_\varepsilon$	$\hat{\sigma}^2_I$	$\hat{\sigma}^2_a$	Model
MD00	-.10	71.5	126.1	294.7	(101)(011)
MN00	-.10	123.1	3.6	180.7	(101)(011)
MN10	0	2807.7	0	145.9	(101)(011)

b. Key SDA Statistics from the MB and MM Methods

Series	$SDSE$	SDA_{MB}	$SDA_{MB} - SDSE$ (%)	SDA_{MM}	$SDA_{MM} - SDSE$ (%)	End-Ctr (%) MB	End-Ctr (%) MM
MD00							
Center	8.46	7.47	-11.6	8.53	+0.8	19.4	0.7
End	8.46	8.92	+5.5	8.58	+1.5		
MN00							
Center	11.09	9.80	-11.6	10.06	-9.3	6.7	-11.7
End	11.09	10.46	-5.8	8.89	-19.9		
MN10							
Center	52.99	46.94	-11.4	46.94	-11.4	-4.4	-5.6
End	52.99	44.86	-15.3	44.30	-16.4		

Table 2. Frequency Table for End Behavior, by Relative Size of Sampling Error

		SDA(end) > SDA(center)		SDA(end) > SDSE		SDA(end)
	Total	MB	MM	MB	MM	MB>MM
DVR						
≤ 1/2	38 (30%)	38 (100%)	11 (29%)	12 (32%)	12 (32%)	26 (68%)
>1/2	90 (70%)	2 (2%)	0 (0%)	0 (0%)	0 (0%)	80 (89%)
Total	128	40 (31%)	11 (9%)	12 (9%)	12 (9%)	106 (83%)

Table 2 shows results for property (1) for 128 series, overall and by DVR size category. Overall, only 31% of the series satisfy (1) with the MB method and a scant 9% with MM. However, with $DVR \le 1/2$, the condition is satisfied for 100% of series with MB and 29% with MM. With $DVR > 1/2$, only two series satisfy the condition and only with the MB method; these series both have $DVR = .51$. Significantly, SDA_{MB} exceeds SDA_{MM} at the ends 68% of the time when DVR $\le$1/2 and 83% overall. The exceptions are cases where a large irregular is estimated. The characteristics observed here are inherent in the measures, not the result of estimation. Previously conducted simulation experiments with the MM method confirm the kinds of shapes noted here; the MB measure comes entirely from models and the X-11 filters.

Figure 2. Scatterplot of Observed Series vs. SE Standard Deviations, MD00

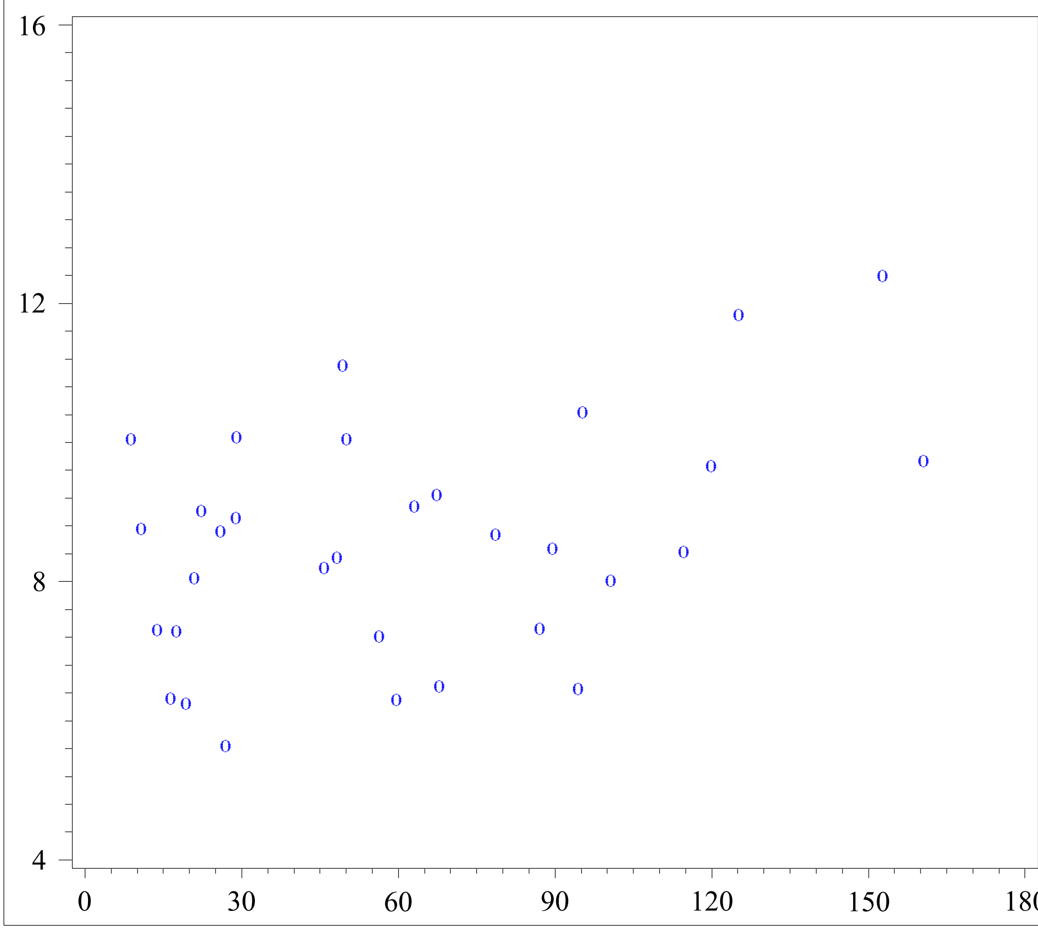
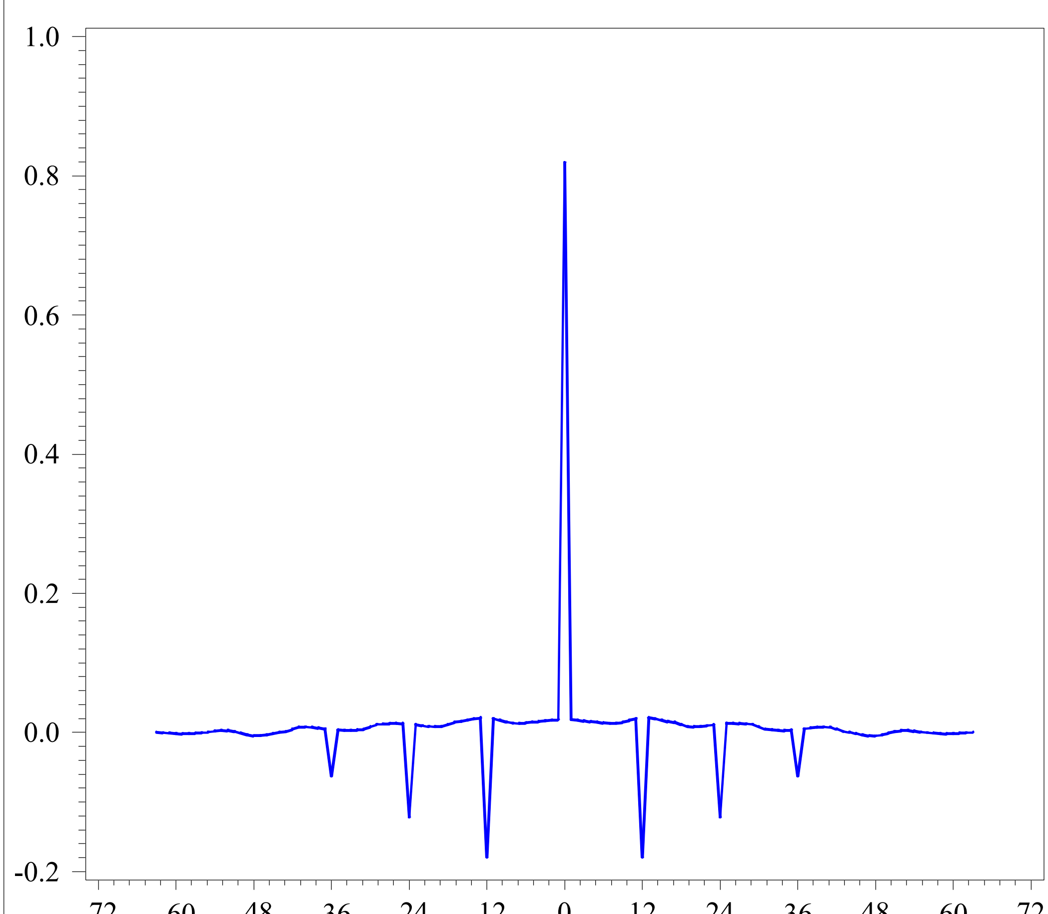


Figure 3. X-11 Seasonal Adjustment Filter, 13-point Henderson, 3x5 Seasonal



The main finding in Table 2 is disappointing. Property (1) fails most of the time. Yet the breakdown by DVR size suggests that when the sampling error is not too large, one or both measures may indeed satisfy (1). Among the 128 press release series are 17 highly aggregated series, including 13 "supersectors" defined under the current industry classification system and Total Private (Sector) Employment, the most highly aggregated series for which SE information is available. Eleven of these series satisfy property (1) with the MB method and 5 with MM. Overall, the empirical results suggest that the MB method is likely to satisfy the intuitive property $SDA(end) > SDA(center)$, as long as the SE contribution is less than half.

3. Key Calculations for the Model-based (MB) Method

This section examines in some detail the matrix calculations for the MB method, in order to put the empirical results of the previous section on a stronger footing. Following the notation and formulas of Bell and Kramer (1999), the equation for the covariance matrix of $Y - \hat{y}$ is

$$Var(Y-\hat{y}) = Var(\varepsilon) + Var\begin{bmatrix} b \\ 0 \\ f \end{bmatrix} - Cov(\begin{bmatrix} b \\ 0 \\ f \end{bmatrix}, \varepsilon) - Cov(\varepsilon, \begin{bmatrix} b \\ 0 \\ f \end{bmatrix})$$

where $b = y_b - \hat{y}_b$ and $f = y_f - \hat{y}_f$, so that the covariance matrix of the seasonal adjustment error $A^* - \hat{A}$ is

$$VARA = Var[\Omega(Y-\hat{y})] = \Omega Var(Y-\hat{y})\Omega` = \Omega Var(\varepsilon)\Omega` + VFA + CVA.$$

In this last equation, VFA denotes the contribution of forecast and backcast errors and CVA the contribution of the covariance term between these errors and sampling error. Let's focus on the diagonal elements. With a stationary model for the sampling error, the first term is constant. Thus, movement in $SDA^2_{MB}(t) = VARA[t,t]$ across time comes entirely from $NET = VFA + CVA$. If the observed series is sufficiently long, the central seasonally adjusted values don't depend on forecasts or backcasts, so the contribution of NET in the center is 0. In particular, this means

( ) $$SDA_{MB}(end) > SDA_{MB}(center) \Leftrightarrow NET(end) > 0.$$

X-11 seasonal adjustment filter matrix Ω

Figure 3 graphs the weights of the X-11 seasonal adjustment filter using the 13-point Henderson trend filter and the "X-11 default" seasonal filter. The latter refers to use of the 3x3 and 3x5 filters in successive iterations of the basic X-11 calculations. The overall filter is symmetric and has length 169. The key weights are 0.82 at the center and -.18, -.12, and -.06 at distances 12, 24, and 36 from the center. For the example in the next section, we take *n*=169 and *m*=84. This means that (1) the central time point *t=85* doesn't depend on backcasts and forecasts and (2) forecasts do not depend on backcasts and vice versa. The matrix Ω is $(n \times n+2m)$; it is convenient to break it down as

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \end{bmatrix},$$

where Ω_1 and Ω_3 are $n \times m$ and Ω_2 is $n \times n$. It is useful to pinpoint where the key weights fall in these matrices. Each row contains a full set of filter weights. The last row of Ω is 0 through column 168, has the left part of the filter in columns 169-252, the central weight .82 in column 253, and the right part in columns 254-337. Table 3 contains the columns where the key weights fall for this last row in Ω, Ω_2, and Ω_3.

Table 3. Large Filter Weights: Column Locations in Row 169 (last row)

Wt.		-.06	-.12	-.18	.82	-.18	-.12	-.06
	Dimension							
Ω	169x337	217	229	241	253	265	277	289
Ω_2	169x169	133	145	157	169	-	-	-
Ω_3	169x84	- - - -	12				24	36

Formulas for variance and covariance terms

We present some key formulas, which are ingredients to understanding behavior of the MB measure. A technical report (Scott, 2010) contains details.

Proposition 1 (Brockwell and Davis, 1991).

Asymptotically, the forecast error covariance matrix is given by

$$VF = Var(f) = \sigma^2 \Psi\Psi`,$$

where Ψ is lower triangular and, starting from the diagonal, columns contain coefficients from $\psi(z) = \theta(z)/\phi(z)$.

Proposition 2.

(i) $f = y_f - \hat{y}_f = C(w_f - \hat{w}_f)$

(ii) $\hat{w}_f = \Sigma_{32}\Sigma_{22}^{-1} w_o$, where $\Sigma_{32} = Cov(w_f, w_o)$, $\Sigma_{33} = Var(w_o)$

(iii) $Cov(f, \varepsilon) = \sigma_\varepsilon^2 \begin{bmatrix} 0 & H & I_m \end{bmatrix}$

Remarks.

1. The matrix C is basically the inverse to a differencing matrix.
2. The matrix H includes a contribution $-CCov(\hat{w}_f, \varepsilon_o)$. From ii), H clearly depends on both the signal and the sampling error, but we claim that $Cov(f, \varepsilon)$ is primarily a function of the sampling error.

Proposition 3.

(i). The forecast error contribution to VARA is $VFA = \Omega_3 VF \Omega_3`$

(ii). The diagonal terms of VFA are $VFA[t,t] = VA1(t) + VA2(t)$, where

$$VA1(t) = \sum_{i=1}^{m} \Omega_3^2[t,i] VF[i,i], \quad VA2(t) = \sum_{i \neq k} \Omega_3[t,i]\Omega_3[t,k] VF[i,k]$$

(iii). A simple approximation at the last time point for the second term is

$$\sigma^{-2} VA2(169) \approx 2\Omega_3[169,12]\left(\Omega_3[169,24] + \Omega_3[169,36]\right) VF[12,24]$$
$$+ 2\Omega_3[169,24]\Omega_3[169,36] VF[24,36]$$

Proposition 4.

(i). The contribution of the covariance term in the forecast period is $CVA = -(CVA1 + CVA2)$, with

$$CVA1 = \Omega_3 H \Omega_2` + (\Omega_3 H \Omega_2`)`, \; CVA2 = 2\sigma_\varepsilon^2 \Omega_3 \Omega_3` = 2\sigma_\varepsilon^2 \sum_{h=1}^{m} \Omega_3^2[t,h]. \; (4)$$

(ii). For $t > n/2$,

$$\Omega_3 H \Omega_2`[t,t] \approx \sum_{j=1}^{3} \Omega_3[t, t-n+12j] \sum_{k=0}^{3} \Omega_2[t, t-12k] H[t-n+12j, t-12k] \qquad (5)$$

In this last formula, we adopt the convention $\Omega_3[t, t-n+12j] = 0$ whenever $t-n+12j < 1$. As t moves backward from 169, the large weights move left in the rows of Ω, which means that fewer terms are nonzero. For $t < 134$, the approximation becomes 0.

4. Characterization of End-behavior – an Example

Using formulas in Section 3, our strategy is to approximate variances or standard deviations of the MB measure in terms of separate contributions of signal and sampling error. We carry out the analysis for an

EXAMPLE. The observed series is $y = Y + \varepsilon$ and the models for the signal Y and the sampling error ε are

$$u_t = (1-B^{12})Y_t = (1-\theta_1 B)(1-\theta_{12}B^{12})a_t, \quad a_t \sim WN(0,\sigma_a^2)$$

$$\varepsilon_t \sim WN(0,\sigma_\varepsilon^2).$$

Parameter values are $\theta_1 = -.2, \theta_{12} = .5, \sigma_a^2 = 100, \sigma_\varepsilon^2 = 64$; in agreement with Section 3, the series length and forecast period length are n=169 and m=84, respectively. These models are close to those selected for EH00, Education & Health Services, one of the supersectors treated in Section 2. The signal model is like an airline model, except that it lacks a first difference. Since the employment data in Section 2 are monthly change in logs, the above model fits when log employment follows an airline model This example does satisfy (1). For the last time point, our variance measure is

$$\begin{aligned} SDA_{MB}^2(169) &= \text{SE contribution}(169) + VFA(169) + CVA(169) \\ &= 50.232 + 18.635 - 17.482 = 51.386 \end{aligned}.$$

and at the end $NET = 18.635 - 17.481 = 1.153 > 0$ as might be expected, since $DVR = 128/258 = .496$. The SDA_{MB} values are 7.09, and 7.17 for the center and end time points, both below the SE standard deviation $\sigma_\varepsilon = 8$.

We now proceed to seek an explanation more directly in terms of the signal and sampling error components, as described in (2). The differenced signal u and the differenced sampling error v have variances 130 and 128, respectively. Autocovariances for *u*, *v* and *w* appear in Table 4 and forecast error variances (or MSE's) in Table 5. As expected, the forecast error variance for w increases with forecast lead. The 'Sum' column in Table 5 provides an approximation for $Var(w_f)$ by summing values of the components. The values increase, like the exact values in the last column, but miss a contribution from $Cov(\hat{w},\varepsilon)$ which is affected by the signal component. The approximation does have two advantages:

(1) it is a simple way to separate component contributions,
(2) it shows us that as the forecast lead increases the variance is increasingly dominated by the signal contribution.

We see that 60% of the total comes from the signal at lead 1, increasing gradually to 80% at lead 84. We now compute NET at any time point past the center of the series as

$$NET = VFA(signal) + VFA(SE) - CVA. \qquad (6)$$

Table 4. Autocovariances for w and Its Components

Lag	u	v	w
0	130	128	258
1	25	0	25
2	0	0	0
⋮			
10	0	0	0
11	-10	0	-10
12	-52	-64	-116
13	-10	0	-10
14	0	0	0

Table 5. Forecast Error MSE's for Selected Leads

Lead	u	v	Sum	w
1	100	64	164	183
12	104	64	168	185
13	129	64	193	211
25	155	64	219	237
37	181	64	245	263
49	207	64	271	288
61	233	64	297	314
73	259	64	323	340
84	260	64	324	341

Since $(VFA\ signal)$ is the dominant forecast variance term and $Cov(f,\varepsilon)$ depends primarily on sampling error, it now becomes apparent that shape can vary considerably according to the relative size of the components.

For the covariance term CVA, we apply Proposition 4. From (5), using the weights information from Table 3 and properties of H,

$$
\begin{aligned}
&CVA1(169)/(2\sigma_\varepsilon^2) \approx \\
&(-.18-.12-.06)(.82\cdot H[12,169]-.18\cdot H[12,157]-.12\cdot H[12,145]-.06\cdot H[12,133]) \\
&=-.36\cdot(.82(-.372)-.18(-.232)-.12(-.146)-.06(-.092))=.0869
\end{aligned}
$$

Based on this approximation the total contribution of $CVA1$ to covariance is $-.174\cdot\sigma_\varepsilon^2$, fairly close to the true value $-.159\cdot\sigma_\varepsilon^2$. The terms 1-11 and 13-23 contribute $+.014\cdot\sigma_\varepsilon^2$. With this correction, the approximation becomes $-.160\cdot\sigma_\varepsilon^2$. Also, $CVA2[169,169]=.114\cdot\sigma_\varepsilon^2$. Thus, from Proposition 4 the total covariance contribution for 169, even using the rough approximation, is

$$CVA(169)=-(2\cdot CVA1+CVA2)=-(.114+.174)\cdot\sigma_\varepsilon^2=-.288\sigma_\varepsilon^2,$$

compared to the true value -.273 σ_ε^2. What is significant is that the weight $\Omega_2[169,169]=.82$, by far the largest magnitude weight in the filter (cf. Figure 3), occurs in the $CVA1$ part of the calculation and nowhere else. In fact, it occurs in $CVA1(t)$ only for the last 12 points in the series. Since the covariance term is negative, this explains why a dip is liable to occur over the last 12 points.

We can further pin down these findings. By factoring out the differenced variance from each term in (6),

$$NET=VFA(signal)+VFA(SE)-CVA=a\sigma_u^2+b\sigma_v^2-c\sigma_v^2,$$

for some coefficients a, b, and c. A little algebra yields a criterion for property (1) in terms of DVR, as defined in (3):

$$NET>0 \Leftrightarrow DVR=\sigma_v^2/\sigma_w^2<a/(a+c-b) \tag{7}$$

For our example, we have $a=.1107$, $b=.0332$, and $c=.1366$, so

$$SDA(end)>SDA(center) \Leftrightarrow DVR<.1107/(.1107+.1366-.0332)=.517.$$

This is satisfied for our example, since DVR=128/258=0.496. As a further check, we carry out the analysis with $\sigma_\varepsilon^2=(8.5)^2=72.25$ and other parameters unchanged, so that DVR=.526. We find $NET=.029$, positive but very close to 0. We see the criterion doesn't quite work. There are approximations involved. We've adjusted the two forecast variance terms to add to the exact value and the last term is not strictly a function of sampling error. Also, a, b, and c change with the disturbance variances. Using values $a = .1118$, $b = .0336$, and $c = .1340$ from the calculations with $\sigma_\varepsilon = 8.5$, the criterion becomes

$$NET(169) > 0 \Leftrightarrow DVR < .1118 / (.1118 + .1340 - .0336) = .527 \text{ ,}$$

which appears very close to being correct. This gives us the sense that (7) works approximately (and maybe iteratively) in identifying when $SDA(end) > SDA(center)$.

Our machinery allows us to analyze the dip at t=169-11=158 as well. At t=158, the signal represents 76% of the forecast variance. Exact calculations yield

$$NET(158) = 12.929 + 4.084 – 17.622 = -.609.$$

We also have the approximation

$$NET(158) = .0995\sigma_u^2 + .0319\sigma_v^2 - .1377\sigma_v^2 \text{ .}$$

From this equation for NET, we find

$$NET(158) > 0 \Leftrightarrow DVR < .0995 / (.0995 + .1377 - .0319) = .485 \text{ .}$$

Computing variances with $\sigma_\varepsilon^2 = 7.75^2 = 60.0625$ (and DVR=.480), we find

$$NET(158) = 12.856 + 3.811 - 16.710 = -.043 \text{ .}$$

From calculations for this case, the refined criterion becomes

$$NET(158) > 0 \Leftrightarrow DVR < .0989 / (.0989 + .1391 - .0317) = .479 \text{ ,}$$

which appears about right. Given the signal model, $SDA_{MB}(t)$ can be expected to stay above SDA_{MB} at the center as long as DVR is below .479, which corresponds closely to $\sigma_\varepsilon^2 < 60$.

We have treated only a particular model form and set of MA parameters. However, the example seems strong enough to make the following characterization of SDA for change, at least when the airline model fits the original series. In moving from the center of the series to the end, the forecast error variance makes an increasing positive contribution to SDA and its size depends increasingly on the signal near the end of the series. The covariance term in SDA_{MB}^2 is negative and depends mostly on the sampling error. Both variance and covariance effects are strongest during the last 12 months. The covariance term CVA jumps sharply at n-11 and is relatively stable up to the end, while the variance term grows slowly but steadily right up to the last time point n. Thus, their net has a minimum at n-11 and tends to increase from there to n. Given signal and SE models, we can find a cutoff value for

(1) SDA_{MB} staying entirely above its central value or nearly so,

(2) $SDA_{MB}(end) > SDA_{MB}(center)$

in terms of the relative size of the SE. DVR, the relative contribution of sampling error to total variance on the differenced scale, is informative. $SDA_{MB}(end) > SDA_{MB}(center)$ corresponds fairly closely to DVR<0.5.

5. Summary

The model-based (MB) and moments-matching (MM) methods both provide conceptually reasonable variance measures for error in seasonal adjustment which account for sampling error. The main advantage of the MB method over the MM method is that it captures more fully the uncertainty at the end of the series, which is the most important point in time. Both methods tend to have unnatural dips in the first and last years of the series. These dips are particularly pronounced when the sampling error is dominant. The model-based (MB) method is likely to provide satisfactory measures of variance, including having end values greater than central values, when the relative contribution of the sampling error is less than half the variance of the differenced series, as measured by the "DVR" statistic, defined in Section 2. Both the large-scale empirical study and the theoretical findings for a typical example support this conclusion.

For the employment change application presented in Section 2, we feel that we have established that the model-based (MB) method is tenable for use in assessing significance of monthly change, as long as the SE contribution is not too large. This is in fact the case for a majority of the highly aggregated series.

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