Properties of Smoothed Design-Based Variance Estimators from Complex Sample Surveys

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Abstract

Due to relatively high levels of sampling variability, direct design-based variance estimators are often smoothed before publication. For example, some federal statistics programs publish the medians of a sequence of monthly direct variance estimates, or functions of these medians. The properties of these smoothed estimators depend on several underlying conditions, including sample size; effective degrees of freedom for the direct estimators; correlation of the direct estimators across months; and temporal patterns in the true variances. We compare and contrast these properties with the corresponding properties of generalized variance function (GVF) estimators.

Key Words: Bias, Design-based inference, Generalized variance function, Median, Model-based inference, Superpopulation model, U.S. Current Employment Statistics Program, Variance estimator stability

1. Introduction

In large-scale periodic sample surveys, standard design-based variance estimators often display substantial variability over periods and among cross-sectional subpopulations. For such cases, a survey organization may need to address the following questions.

1. To what extent can the observed variability be attributed to, respectively:
   (a) Sampling variability of the variance estimator itself
   (b) Observable factors (e.g., the number of respondents in the specified time periods) that may have a direct effect on the true design variance
   (c) Other factors that are not readily observable but that may nonetheless have a substantial effect on the true design variance
2. Based on answers to the questions in (1):
   (a) What are some appropriate methods for "smoothing" or otherwise combining information from variance estimators over multiple periods?
   (b) What are appropriate inferential uses of the resulting variance estimators?

To address the questions in areas 1 and 2, this paper will use the following notation. Let θ_{jt} be a finite-population parameter for domain j and period $t, j = 1, \ldots J; t = 1, \ldots T$; let $\hat{\theta}_{jt}$ be the corresponding design-based point estimator; and define $V_{jt} = V_p(\hat{\theta}_{jt})$, where $V_p(\cdot)$ represents variance evaluated with respect to the sample design. In addition, let $V_{jt}^{\dagger}$ represent a general estimator of the design variance V_{jt}. The remainder of this paper will consider several classes of estimators $V_{jt}^{\dagger}$, including direct design-based variance estimators

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$\hat{V}_{jt}$; temporal medians of the direct estimators $\tilde{V}_{jt}$; temporal means of the direct estimators $\bar{V}_{jt}$; and more complex estimators based on generalized variance functions V^*_{jt}. For any estimator $V^{\dagger}_{jt}$, we will consider three evaluation criteria:

A. The expectation of $V^{\dagger}_{jt}$. We generally prefer to use estimators $V^{\dagger}_{jt}$ that are approximately unbiased for the true design variance V_{jt}

B. The stability of $V^{\dagger}_{jt}$. We generally prefer to use estimators $V^{\dagger}_{jt}$ that have relatively small variances. We often will characterize the stability of $V^{\dagger}_{jt}$ through a Satterthwaite-type "degrees of freedom" term $d^{\dagger}_{jt} = \left\{V_p(V^{\dagger}_{jt})\right\}^{-1} 2\,(V_{jt})^2$

C. Performance of the associated confidence intervals $\hat{\theta}_{jt} \pm t_{d^{\dagger}_{jt},1-\frac{\alpha}{2}}\,(V^{\dagger}_{jt})^{\frac{1}{2}}$ where $t_{d^{\dagger}_{jt},1-\frac{\alpha}{2}}$ is a customary $\left(1-\frac{\alpha}{2}\right)$ quantile of a t distribution on $d^{\dagger}_{jt}$ degrees of freedom. We generally prefer estimators $V^{\dagger}_{jt}$ that lead to confidence intervals that have true coverage rates greater than or equal their nominal rates $1-\alpha$; and that have relatively small widths.

For example, criteria (A) through (C) are of interest for the Current Employment Statistics (CES) of the U.S. Bureau of Labor Statistics. The CES survey collects data on employment, hours, and earnings from 390,000 nonfarm establishments monthly. Employment is the total number of persons employed full or part time in a nonfarm establishment during a specified payroll period. An establishment is defined to be an economic unit, generally located at a single place, which is engaged predominantly in one type of economic activity. The CES sample design uses stratified sampling of unemplyment insurance (UI) accounts with strata defined by state, industry and employment size class (BLS Handbook, 2011). The primary CES design goal is to meet the precision requirements specified for the national estimates. However, within some domains, effective sample sizes become so small that the standard design based variance estimators are not precise enough to satisfy the needs of prospective data users (Eltinge, Fields, Fisher, Gershunskaya, Getz, Huff, Tiller and Waddington, 2001; and Gershunskaya and Lahiri, 2005). It is necessary to have stable estimators of V_{jt} for the finer domains. At present, for a given domain, the CES program publishes standard errors based on three-year averages of the temporal medians of the estimators, $\hat{V}_{jt}$, where $\hat{V}_{jt}$ is based on balanced repeated replication with Fay factors; and for a given year the monthly indices t cover the six months of April through September.

The remainder of this paper will focus on estimation and inference issues motivated by the abovementioned CES application. However, the general ideas considered here are potentially applicable in settings beyond the CES. For example, the U.S. Consumer Price Index (CPI) publishes measures of sampling variability based on medians of standard errors computed for each of twelve consecutive months. For the CPI, principal interest in variance estimation centers on the variances for the estimated index itself; one-month relative change in the estimated index; three-month change; and twelve-month change. For each of these CPI cases, estimation at the national level and for four geographical regions are of interest. In addition, Jang et al. (2006) considered the use of median design effects for the GVF formulation with the 2003 Scientists and Engineers Statistical Data System (SESTAT) data.

2. Notation and Models

Let $\hat{V}_{jt}$ be a standard design-based estimator of $V_p(\hat{\theta}_{jt})$, and define the estimation error

$$\epsilon_{jt} = \hat{V}_{jt} - V_{jt}. \tag{1}$$

Question (1.a) centers on the error terms ϵ_{jt}. In some applications, $V_{jt}^{-1}\,\hat{V}_{jt}\,d_{jt}$ is distributed as a chi-square random variable on d_{jt} degrees of freedom, where d_{jt} is a fixed term. In that case, ϵ_{jt} has a mean equal to 0 and a variance equal to $d_{jt}^{-1}\,2\,V_{jt}^{2}$.

To address questions (1.b) and (1.c), consider the model

$$log(V_{jt}) = X_{jt}\gamma + q_{jt} \tag{2}$$

where X_{jt} is a B-dimensional row vector of observable predictors; γ is a B-dimensional column vector of coefficients; q_{jt} is a random "equation error" term with mean equal to zero; and $log(\cdot)$ is the natural logarithmic transformation.

An expanded version of model (2) is

$$log(V_{jt}) = \gamma_0 + X_{j\cdot}^{*}\gamma_1 + X_{\cdot t}^{*}\gamma_2 + X_{jt}^{*}\gamma_3 + q_{jt} \tag{3}$$

where we partition the full $JT \times B$ matrix of predictors

$$X = (\underline{\mathbf{1}},\, X_{Dom\cdot}^{*},\, X_{\cdot Time}^{*},\, X_{Dom\,Time}^{*})\,; \tag{4}$$

$\underline{\mathbf{1}}$ is the $(J \times T) \times 1$ vector of ones; $X_{Dom\cdot}^{*}$ is the submatrix of columns of X that depend on the domain j but not the time t; $X_{\cdot Time}^{*}$ is the submatrix of columns of X that depend on the time t but not on j; and $X_{Dom\,Time}^{*}$ is the submatrix of columns of X that depend on both j and t.

In general, we may wish to use model (3) to suggest ways in which combine variance information across grouping variables for which the V_{jt} differences are relatively small. For example, if $\gamma_1, \gamma_2, \gamma_3$ and the variance of q_{jt} were all approximately equal to zero, then all $log(V_{jt})$ terms would be approximately equal to γ_0 and we may wish to combine all of our variance information through a single mean or median. Similarly, if γ_2, γ_3 and the variance of q_{jt} were all approximately equal to zero, and if γ_1 were nonzero, then one may choose to combine information across time, but not across domains. For this case, two potential estimators are the temporal mean and the temporal median.

Relatively simple versions of the domain-specific temporal mean and median based variance estimators are

$$\bar{V}_{j\cdot} = M^{-1} \sum_{t=t_0+1}^{t_0+M} \hat{V}_{jt}$$

and

$$\tilde{V}_{j\cdot} = c_{Md}\; median\left\{\hat{V}_{jt},\ t = t_0+1, t_0+2, \ldots, t_0+M\right\} \tag{5}$$

respectively, where M is the total number of months used in computation of the median; and c_{Md} is a multiplier intended to ensure that $\tilde{V}_{j\cdot}$ is approximately unbiased for the corresponding true variance V_{jt}.

Conversely, if γ_1, γ_3 and the variance of q_{jt} were all approximately zero, and γ_2 were nonzero, then one may choose to combine information across domains but not time. Use of medians or other outlier-resistant estimation methods may be of special interest when individual variance estimators $\hat{V}_{jt}$ are at risk of taking on relatively extreme values.

Finally, consider cases in which the errors ϵ_{jt} are substantial; the equation errors q_{jt} are relatively small; and the regression terms $X_{jt}\gamma$ may vary substantially with respect to j or t. For those cases, it may be appropriate to use a generalized variance function model to "smooth" the variance estimators across time periods. See, e.g., Johnson and King (1987), Valliant (1987), O'Malley and Zaslavsky (2005), Wolter (2007) and Cho et al. (2012a) for some background on generalized variance functions.

3. Temporal and Cross-Sectional Means and Medians

3.1 Properties of smoothed variance estimators

In evaluation of the properties of $\tilde{V}_{j\cdot}$, for example, as an estimator of the true design variance V_{jt} for a given month, principal attention centers on the following

A. The design or design-model expectation, variance and approximate distribution of $\tilde{V}_{j\cdot} - V_{jt}$. In particular, define the following terms

- **(i)** Let $c_{Mdt}^{-1} = V_{jt}^{-1}\, E_p\left(\tilde{V}_{j\cdot}\right)$.

  If the abovementioned median were design-unbiased for V_{jt}, then $c_{Mdt} = 1$. Thus, deviation of c_{Mdt} from 1 provides an indication of the design bias of the simple temporal median as an estimator of the month-specific variance V_{jt}.

- **(ii)** Let $b_{Mdt} = V_{jt}^{-2}\, V_p\left(\tilde{V}_{j\cdot}\right)$, the relative variance of the temporal median as an estimator of V_{jt}.

- **(iii)** In addition, define

$$\begin{aligned} \delta_{Mdt} &= 2\, c_{Mdt}^{-2}\, b_{Mdt}^{-1} \\ &= 2\left[E_p\left(\tilde{V}_{j\cdot}\right)\right]^2 \left[V_p\left(\tilde{V}_{j\cdot}\right)\right]^{-1}. \end{aligned} \tag{6}$$

  Under Satterthwaite-type approaches and regularity conditions, the expression

$$V_{jt}^{-1}\, c_{mdt}\; \tilde{V}_{j\cdot}\; \delta_{Mdt}$$

  may follow approximately a chi-square distribution on δ_{Mdt} degrees of freedom.

B. The properties considered in A generally will depend on:

- **(i)** the distribution of the sampling errors ϵ_{jt}, including their variances and their temporal correlations;
- **(ii)** the mean structure for V_{jt} reflected in the regression term $X_{jt}\gamma$ from model (2);
- **(iii)** the distribution of the equation errors q_{jt}; and

for the current discussion, we assume that the true variances are constant across months, i.e.,

$$V_{jt} = V_{j\cdot} \qquad \forall t = t_0 + 1,\, t_0 + 2, \ldots, t_0 + M. \tag{7}$$

3.2 Simulation results

Table 1 presents population medians of a chi-square distribution on d degrees of freedom divided by d. It shows that medians have a bias especially for cases that involve relatively small values of d. Consequently, for cases in which unbiased estimation of V_{jt} is important, one would need to adjust the temporal medians.

We will now examine properties of median and mean by comparing their expectation, variance and degrees of freedom values. Table 2 presents the sample median and mean of M independent χ^2_d/d random variables for specified values of M and d. Each row is based on $10,000$ replications. The term δ_{Md} provides the approximate "degrees of freedom" term attributable to the sample median and mean. Note that in each case, $d < \delta_{Md} < M \cdot d$ for the sample median, and $d < \delta_{Md} \approx M \cdot d$ for the sample mean. In other words, under idealized χ^2_d conditions, $\tilde{V}$ is less stable than $\bar{V}$.

We also examined two cases in which variance estimator values are correlated across months. Table 3 presents properties of the sample median and mean of M consecutive diagonal elements from a $Wishart_d(V(\rho))$ random matrix, where $M = 6, d = 6$ and $V(\rho)$ is an $M \times M$ equicorrelation matrix with off-diagonal elements equal to the specified value of ρ. Note especially that for each value of $\rho = 0.1$ through 0.9, the resulting $\delta_{Md\rho}$ values of the median and mean are less than the values of δ_{66} in Table 2 for the independent-observation cases. Similarly, Table 4 presents properties of the sample median of M consecutive diagonal elements from a $Wishart_d(V_{AR}(\rho))$ random matrix where $M = 6, d = 6$ and $V_{AR}(\rho)$ is an $M \times M$ correlation matrix for a first-order autoregressive model with autoregressive parameter ρ.

We further examined cases in which values are from different distributions, chi-square or Wishart. Table 5 presents properties of the sample median of M consecutive diagonal elements of a lognormal random vector with first and second moments constrained to match the first and second moments of a $Wishart_d(V(\rho))$ distribution as specified for Table 3. Note that the resulting approximate "degrees of freedom" terms δ^*_{Md} are considerably less than the corresponding $\delta_{Md\rho}$ terms for the median reported in Table 3, and are slightly less than the corresponding $\delta^*_{Md\rho}$ in Table 4.

Table 6 presents confidence interval properties after adjusting median values and adjusting degrees of freedom for each estimator, $\hat{V}$, $\bar{V}$ and $\tilde{V}$. Median values are adjusted by multiplying c_{Mdt} (discussed in Section 3.1) to ensure that $\tilde{V}_{j\cdot}$ is approximately unbiased for the corresponding true variance V_{jt}. For the independent-observation case, all three have coverage rates close to nominal level. $\bar{V}$ has the smallest mean-width and inter-quartile range (IQR) while the bias-adjusted $\tilde{V}$ is slightly less efficient as measured by width and IQR.

4. Graphical Comparison of the Relative Effects of ϵ_{jt}, q_{jt} and $X_{jt}\gamma$

4.1 Error effect with approximately constant $E(V_{jt})$ and no temporal correlation of error terms

To explore the competing effects of sampling errors, equation errors and GVF mean structure, we consider several hypothetical cases.

Case 1: $E(V_{jt})$ is constant, $V(q_{jt}) \doteq 0$, and M is large.
In this case, the GVF estimator V^*_{jt} and the median-based estimator $\tilde{V}_{jt}$ are both almost identical to the true variance V_{jt}, while the direct variance estimator $\hat{V}_{jt}$ may differ substantially from V_{jt} if $V(\epsilon_{jt})$ is not small. For this case, either V^*_{jt} or $\tilde{V}_{jt}$ may be considered a satisfactory estimator of V_{jt}.

Case 2: $E(V_{jt})$ is constant and $V(q_{jt})$ is nonzero.
Then for sufficiently large M, V^*_{jt} and $\tilde{V}_{jt}$ are approximately equal to each other, but their properties as predictors of the true V_{jt} will depend on the relative magnitudes of $V(\epsilon_{jt})$ and $V(q_{jt})$.
Case 2a: $V(\epsilon_{jt}) << V(q_{jt})$.
Then one generally prefers to use the direct estimator $\hat{V}_{jt}$ instead of either V^*_{jt} or $\tilde{V}_{jt}$, which are essentially oversmoothed estimators.
Case 2b: $V(\epsilon_{jt}) >> V(q_{jt}) > 0$ and M is small.

Then the GVF estimator V^*_{jt} may be an imperfect predictor of the true design variance V_{jt}, but still may be preferable to $\tilde{V}_{jt}$, which has a larger error as a predictor for V_{jt}.
Case 2c: Both $V(\epsilon_{jt})$ and $V(q_{jt})$ are large, and M is moderate or small.
Then it is possible that none of V^*_{jt} nor $\tilde{V}_{jt}$ nor $\hat{V}_{jt}$ are satisfactory predictors for V_{jt}.

4.2 Error effect with approximately constant $E(V_{jt})$ and temporal correlation of error terms

Due to the use of rotation samples and estimators that combine data from several consecutive periods, the estimation errors ϵ_{jt} may be correlated over time. In addition, changes in population conditions may lead to temporal correlation of the q_{jt} terms. For example, changes in the economic cycle may lead to inflation or deflation in the true V_{jt} that is not captured by the predictors X_{jt} used in our GVF model (2), and these changes may persist over a substantial number of periods.
Case 3: Estimation error terms ϵ_{jt} with strong temporal correlation and M is small or moderate.
Case 3a: Zero temporal correlation in q_{jt} and $V(q_{jt})$ is close to zero and small relative to $V(\tilde{V}_{jt})$. Then the GVF V^*_{jt} is clearly preferable to $\tilde{V}_{jt}$.
Case 3b: Zero temporal correlation in q_{jt}; and $V(q_{jt})$ is greater than zero and not small relative to $V(\tilde{V}_{jt})$. As with case 2c, it is possible that neither $\hat{V}_{jt}$ nor $\tilde{V}_{jt}$ may be satisfactory and choices among use of $\hat{V}_{jt}$ nor V^*_{jt} or $\tilde{V}_{jt}$ would depend on assessment of the relative magnitudes of $V(\epsilon_{jt}), V(q_{jt})$ and $Corr(\epsilon_{j,t-1}, \epsilon_{jt})$.

4.3 Error effect with changing $E(V_{jt})$

Now consider the case in which $E(V_{jt})$ is not constant. To simplify this discussion, we assume $V(q_{jt}) = 0$.
Case 4: Assume that V_{jt} increases substantially over t (i.e., the true variance increases as the reference period moves away from the benchmark period)
Case 4a: Assume that $\{V(\epsilon_{jt})\}^{\frac{1}{2}}$ is small relative to the changes in the true V_{jt}. Then either direct estimator $\hat{V}_{jt}$ or the GVF estimator V^*_{jt} may be satisfactory estimators of the true V_{jt}, but $\tilde{V}_{jt}$ will generally be unsatisfactory because it fails to reflect the important time trends in the true V_{jt}.
Case 4b: Assume that $\{V(\epsilon_{jt})\}^{\frac{1}{2}}$ is large relative to the changes in the true V_{jt}. Then the GVF estimator V^*_{jt} may be more satisfactory estimators of the true V_{jt} than direct estimator $\hat{V}_{jt}$. $\tilde{V}_{jt}$ will generally be unsatisfactory because it fails to reflect the important time trends in the true V_{jt}.

5. Illustration with Variance Estimates from the Current Employment Statistics Program

Sections 4.1 through 4.3 illustrated the hypothetical Cases 1 through 4 defined by the relative magnitudes of $V(\epsilon_{jt})$, $V(q_{jt})$, $Corr(\epsilon_{j,t-1}, \epsilon_{jt})$ and variability of $X_{jt}\gamma$. Several of these cases correspond to empirical results obtained for the Current Employment Statistics Program. For example, Cho et al. (2012b) explored variance function models for data from the years 2005 through 2010. Their empirical results indicated that a log-linear model (8) provides a satisfactory fit for point estimators of population totals for employment in a given ”supersector” industry.

$$log(V_{jt}) = \gamma_0 + \gamma_1 ln(n_{jt}) + \gamma_2 ln(t) + q_{jt} \tag{8}$$

In addition, the errors ϵ_{jt} may be strongly correlated due to the forms of the population total estimator and CES rotation sample pattern, which tended to use most of the same sample units across consecutive months. $V(\epsilon_{jt})$ was computed using balanced half-sample methods with Fay factors, per Judkins (1990).

Figures 9 and 10 present temporal plots of the direct variance estimator $\hat{V}_{jt}$, the GVF based estimator V^*_{jt}, the mean based estimator $\bar{V}_{jt}$, and the median based estimator $\tilde{V}_{jt}$.

Month 1 corresponds to March of 2009, which is the benchmark month; Month 2 corresponds to April 2009; and Month 19 corresponds to September 2010. Note that the mean and median estimators are computed from the direct estimators for the six months, April through September 2010 (Month14-Month19). In addition, the coefficients γ used to compute V^*_{jt} are based on $\hat{V}_{jt}$ data from 2008-2010 across all 14 supersector industry groups. For the total employment, median values are always larger than mean values for all industries. It is because the direct variance estimator of total employment is increasing with respect to the month, i.e., variance increases as it gets farther away from the benchmark month. However, for one-month change and one-month relative which are not presented here, median values are always smaller than mean values. That is because temporal trends with respect to months have been removed in cases of one-month change and one-month relative, and also because of variance estimators have skewed distributions.

Figure 9 presents results of variance estimators of total employment for the durable manufacturing industry in 2009. Note that it does not exhibit any strong evidence of outlying $\hat{V}_{jt}$ values. Figure 10 presents results of variance estimators of total employment for the wholesale trade industry in 2009. Note that the $\hat{V}_{jt}$ value for April is exceptionally small relative to the fitted GVF estimate V^*_{jt} for that month. Thus, Figure 10 describes a case in which one may wish to consider outlier-resistant methods for estimation of a generalized variance function model. Examples of such methods would include M-estimation and outlier-resistant versions of kernel smoothing. Application of these methods will be considered in future work.

6. Summary and Future Work

This paper has explored some properties of temporal medians and means of standard design based on variance estimators.

First, the current simulation work has focused on cases in which scaled versions of the underlying design-based estimators follow standard distributions like the chi-square, Wishart or lognormal distributions. However, temporal medians and other outlier-resistant estimators are of special interest for cases in which the direct design-based estimators follow heavy-tailed distributions like a contaminated lognormal distribution. Consequently, it would be useful to extend the current simulation work to explore the properties of the temporal mean or temporal median of $\hat{V}_{jt}$ under heavy-tailed conditions.

Second, the current simulation work was based on the assumption that the true variances V_{jt} were constant over time. For many cases, the CES and other survey applications, the true variances V_{jt} are not constant over time. Consequently, it is of interest to develop methods for estimation of generalized variance functions that can account for non-constant true variances, and that are also relatively robust against the presence of outliers. This would involve extensions of previous literature on analysis of complex survey data in the presence of outliers, e.g., Chambers (1986), Zaslavsky et al. (2001), Beaumont and Rivest (2009) and references cited therein.

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Table 1: Population medians of a chi-square distribution on d degrees of freedom divided by d

d	1	2	3	4	5	6	7	8	9	10	15	20
Median	0.45	0.69	0.79	0.84	0.87	0.89	0.91	0.92	0.93	0.93	0.96	0.97

Table 2: Properties of the sample median and the sample mean of M independent χ^2_d/d random variables

d	M	**Median**			**Mean**		
		Mean	Variance	δ_{Md}	Mean	Variance	δ_{Md}
2	3	0.80	0.29	4.37	0.94	0.23	6.03
2	6	0.78	0.16	7.82	1.00	0.17	11.98
2	12	0.74	0.08	13.54	1.00	0.08	24.35
6	3	0.94	0.14	12.74	1.00	0.11	17.83
6	6	0.92	0.06	26.34	1.00	0.06	36.16
6	12	0.91	0.03	47.19	1.00	0.03	73.15
10	3	0.96	0.09	21.59	1.00	0.07	29.78
10	6	0.95	0.04	45.10	1.00	0.03	60.37
10	12	0.94	0.02	80.63	1.00	0.02	121.41

Table 3: Properties of the sample median and mean of M consecutive diagonal elements from a $Wishart_d(V(\rho))$ random matrix, where $M = 6, d = 6$ and $V(\rho)$ is an $M \times M$ equicorrelation matrix with off-diagonal elements equal to the specified value of ρ.

ρ	**Median**			**Mean**		
	Mean	Variance	$\delta_{Md\rho}$	Mean	Variance	$\delta_{Md\rho}$
0.1	0.93	0.07	26.12	1.00	0.06	34.75
0.2	0.93	0.08	22.70	1.00	0.07	30.02
0.3	0.94	0.09	19.44	1.00	0.08	24.60
0.4	0.95	0.11	16.13	1.01	0.10	19.61
0.5	0.95	0.13	13.48	1.00	0.12	16.04
0.9	0.99	0.29	6.79	1.00	0.29	7.02

Table 4: Properties of the sample median of M consecutive diagonal elements from a $Wishart_d(V_{AR}(\rho))$ random matrix where $M = 6, d = 6$ and $V_{AR}(\rho)$ is an $M \times M$ correlation matrix for a first-order autoregressive model with autoregressive parameter ρ.

ρ	Mean	Variance	δ^*_{Md}
0.1	0.89	0.07	22.07
0.2	0.89	0.07	21.71
0.3	0.89	0.08	20.44
0.4	0.89	0.09	18.57
0.5	0.90	0.10	16.45
0.9	0.98	0.32	6.09

Table 5: Properties of the sample median of M consecutive diagonal elements of a lognormal random vector with first and second moments constrained to match the first and second moments of a $Wishart_d(V(\rho))$ distribution as specified for Table 3.

ρ	Mean	Variance	$\delta^*_{Md\rho}$
0.1	0.88	0.07	21.25
0.2	0.89	0.08	20.23
0.3	0.90	0.09	17.50
0.4	0.91	0.11	14.72
0.5	0.91	0.14	12.07
0.9	0.99	0.36	5.41

Table 6: Properties of nominal 95% confidence Interval Properties after adjusting median values and adjusting degrees of freedom for each estimator

$V^\dagger$	*df*	*Coverage Rate*	*Mean Width*	*Quantiles of Widths* 0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
$\hat{V}$	6	0.9496	4.69	1.85	2.54	2.97	3.70	4.61	5.57	6.52	7.09	8.28
$\bar{V}$	36	0.9528	4.03	2.96	3.26	3.42	3.71	4.02	4.34	4.64	4.83	5.18
$\tilde{V}$	26	0.9526	4.07	2.80	3.16	3.37	3.69	4.06	4.45	4.79	5.02	5.44

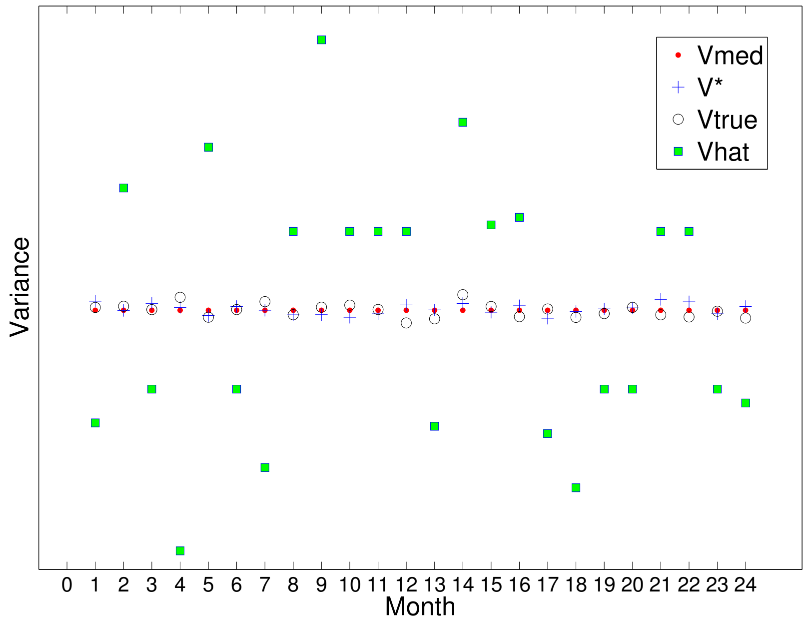

Figure 1: Case 1

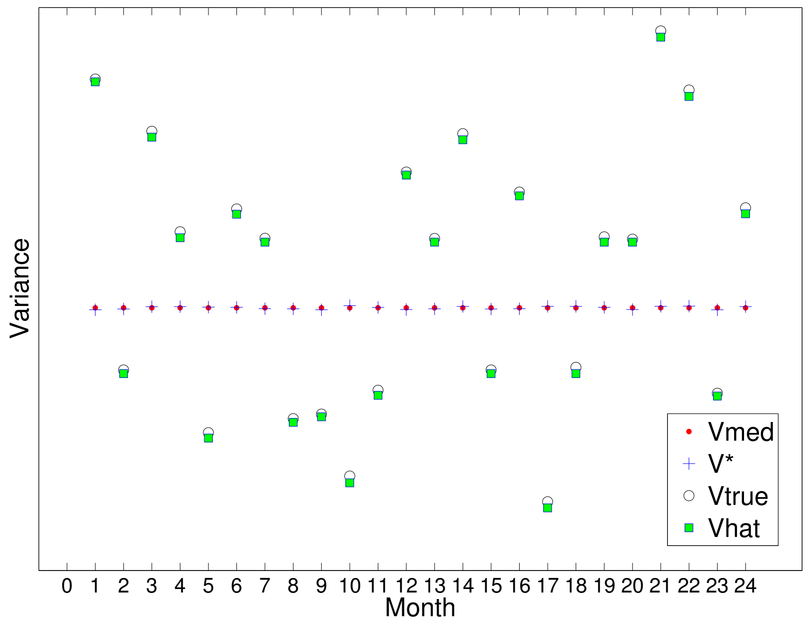

Figure 2: Case 2a

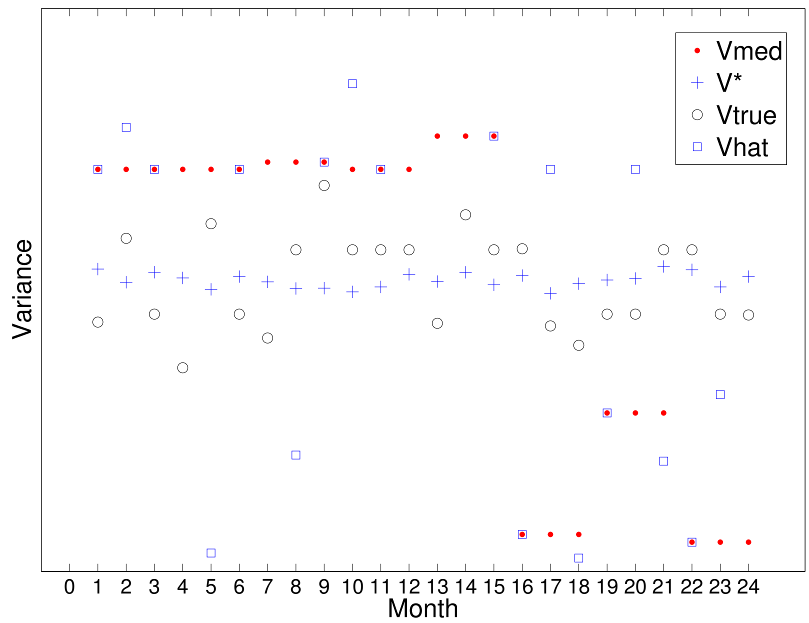

Figure 3: Case 2b

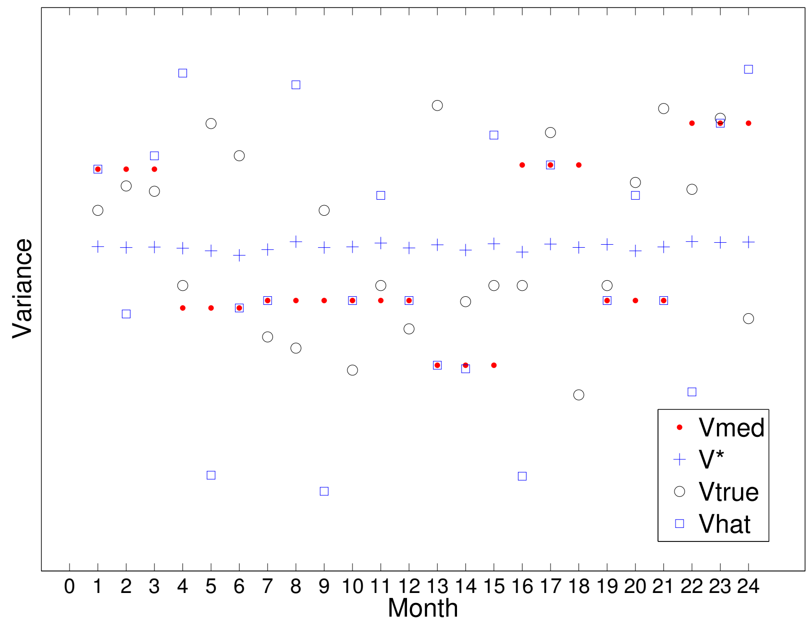

Figure 4: Case 2c

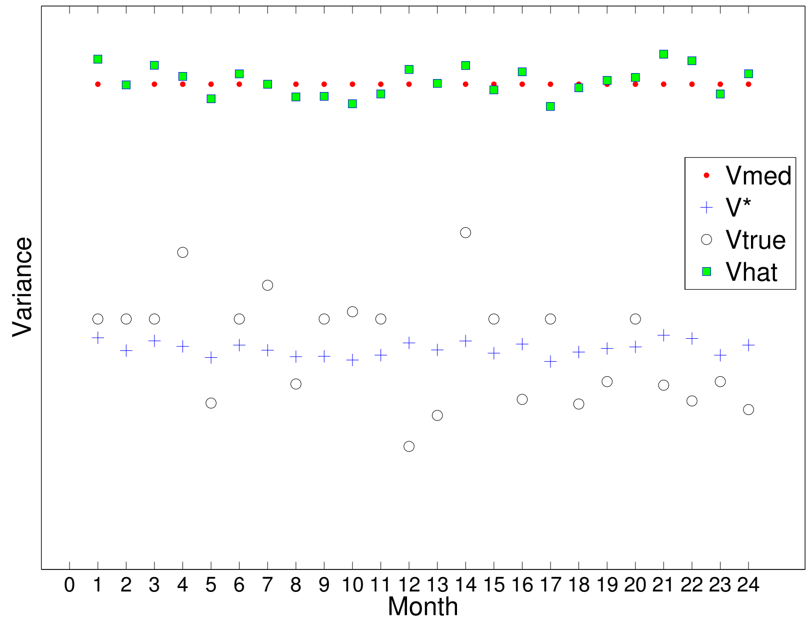

Figure 5: Case 3a

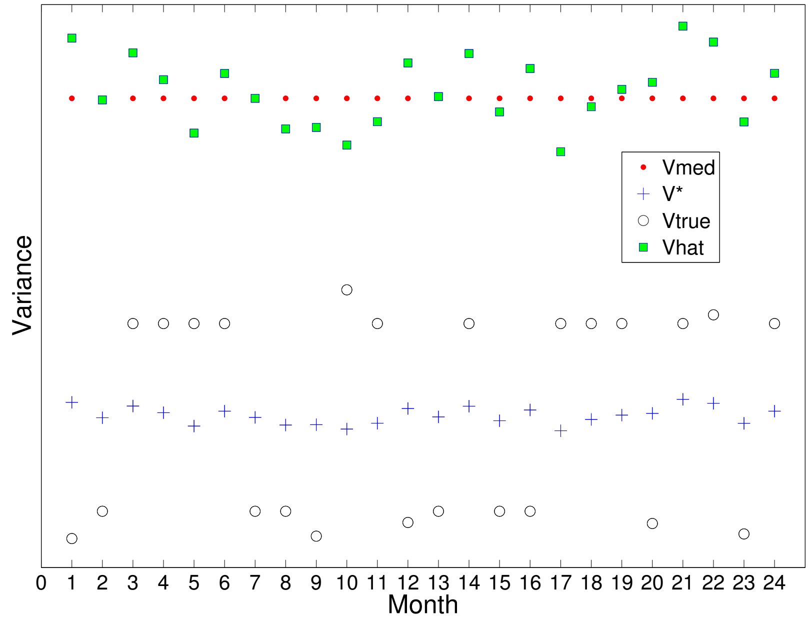

Figure 6: Case 3b

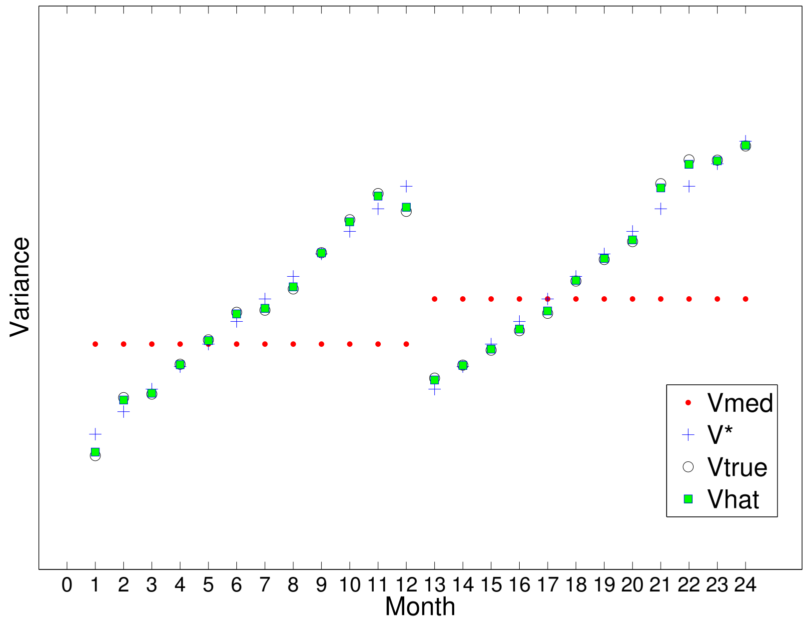

Figure 7: Case 4a

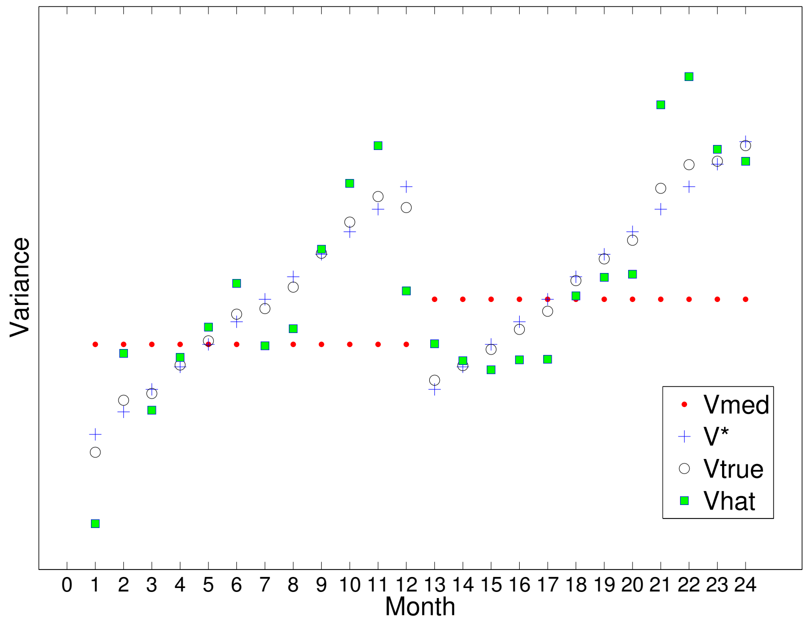

Figure 8: Case 4b

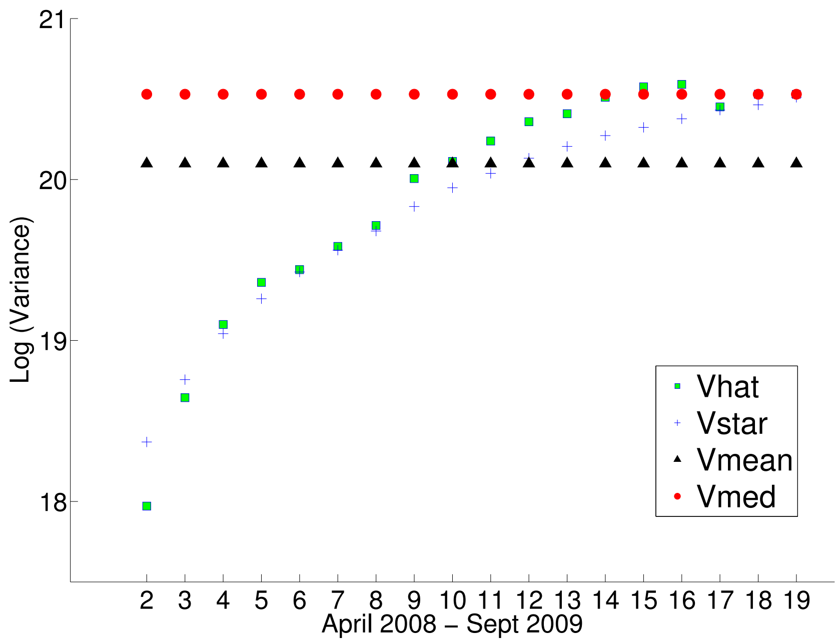

Figure 9: Log (Variance Estimators) of Total Employment for Durable Manufacturing

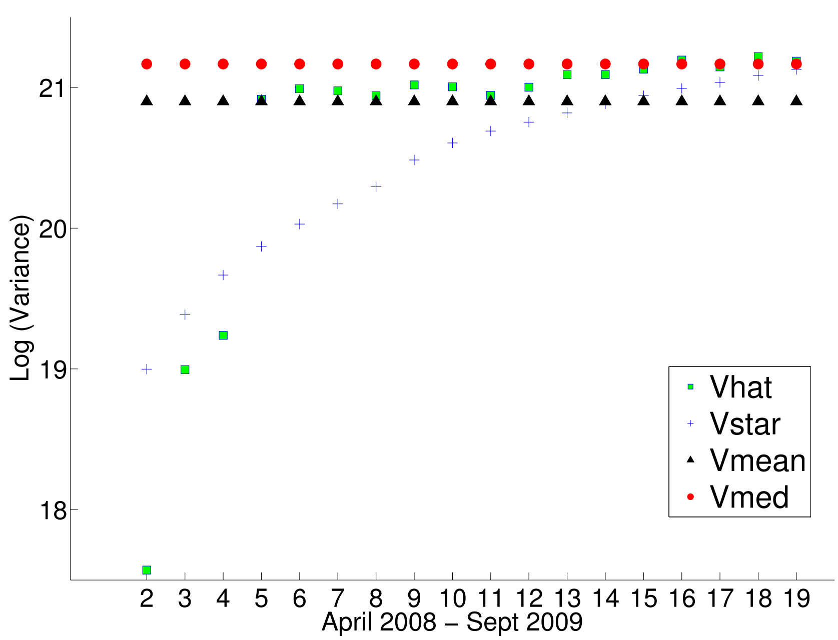

Figure 10: Log (Variance Estimators) of Total Employment for Wholesale Trade