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RESTRICTING REGRESSION SLOPES IN THE ERRORS-IN-VARIABLES MODEL BY BOUNDING THE ERROR CORRELATION

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1. INTRODUCTION

REMEDIES FOR THE ERRORS-IN-VARIABLES problem often take the form of consistently estimable bounds on a parameter, the advantage of such remedies being that they require weaker assumptions than those needed for consistent estimation of the parameter itself. The seminal result is the "errors-in-variables bound" of Gini (1921), which states that the slope on the mismeasured variable lies between the probability limit of the least squares estimate of the coefficient on the proxy and the probability limit of the "reverse" regression estimate of the same coefficient. This result has been generalized to multiple mismeasured regressors by Kalman (1982) and Klepper and Leamer (1984), and to equation systems by Leamer (1987).

All these results require that the measurement error(s) be uncorrelated with the equation error(s).[2] It is not hard to think of examples where these errors are correlated, however. The consumption function study of Friedman (1957) assumed zero correlation between an equation error interpreted as transistory consumption and a measurement error interpreted as transitory income. But imperfect capital markets induce correlation by interfering with intertemporal consumption smoothing, and some determinants of transitory income, such as severe illness, affect transitory consumption as well. Another example is an earnings equation where education is quantified in terms of quality-adjusted years of schooling, but is measured by actual years of schooling. If "ability" is an omitted variable then the measurement and equation errors will be negatively correlated. This is because within any group of individuals having identical values for "true" education, persons of above average ability will tend to have above average earnings but, since they also tend to learn faster, below average years of schooling.

No bounds exist if the zero-correlation assumption is dropped, a point established in Krasker and Pratt (1986), Bekker, Kapteyn, and Wansbeek (1987), and Erickson (1989). Generalizing the Gini bound to such situations therefore requires alternative prior information. Krasker and Pratt use a prior lower bound on the correlation between the proxy and the true regressor, and derive values for this bound ensuring that regression coefficients in the true model have the same signs as the corresponding coefficients in the equation with the proxy. Bekker, Kapteyn, and Wansbeek derive finite bounds, using as their prior input an upper bound on the covariance matrix of the errors.

It is likely that individuals who believe the equation error-measurement error correlation is nonzero also believe at least as strongly that it is neither -1 nor 1. The present paper works out the implications of placing upper and lower bounds on this correlation in a multiple regression model with exactly one mismeasured regressor.[3] Letting ρ denote the error correlation, r denote the population partial correlation between the

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[2] In this paper "equation error" means the sum of a true equation error and any measurement error in the dependent variable. "Measurement error" refers only to measurement error in a regressor.

[3] Allowing only one mismeasured regressor still permits wide application. Recent papers that explicitly assume only one mismeasured regressor include Barro and Sala-i-Martin (1992), and Solon (1992).

dependent variable and proxy, and U and L denote the prior bounds, the main results are as follows, for the case where r is positive: if $L \leqslant r \leqslant U$, then the coefficient is unrestricted; if $U < r$, then the coefficient on the unobserved true regressor lies in a finite interval of positive numbers; and if $L > r$, then, surprisingly, the coefficient can be any number *not* in the Gini interval. In each case the set of possible values for the coefficient on a correctly measured regressor is the image, under a linear mapping, of the possible values for the coefficient on the unobserved regressor. To reduce the cost of assessing prior bounds, and to assist public reporting, corollary results are given that summarize those (U, L) combinations ensuring any desired coefficient satisfies any hypothesized inequality.

The paper is arranged as follows: Section 2 presents the model, and determines when prior bounds on ρ imply restrictions on the coefficient of the proxied regressor. Section 3 reports restrictions on the remaining coefficients. An appendix contains the proof to Theorem 1. Corollaries 1.1 and 1.2 have obvious proofs which are left to the reader.

2. IMPLICATIONS OF PRIOR BOUNDS ON THE ERROR CORRELATION

The multiple linear regression model with one mismeasured regressor can be written

$$y_i = \gamma + \chi_i\theta + Z_i\delta + u_i, \tag{1}$$

$$x_i = \chi_i + \varepsilon_i, \tag{2}$$

where only the scalars y_i and x_i and the $1 \times k$ vector Z_i are observable. It is assumed that $\{\chi_i, Z_i, u_i, \varepsilon_i\}$ is an independent random sequence with covariance matrix

$$\operatorname{var}(\chi_i, Z_i, u_i, \varepsilon_i) = \begin{pmatrix} \sigma_{\chi\chi} & \sigma_{\chi z} & 0 & 0 \\ \sigma_{z\chi} & \sigma_{zz} & 0 & 0 \\ 0 & 0 & \sigma_{uu} & \sigma_{u\varepsilon} \\ 0 & 0 & \sigma_{\varepsilon u} & \sigma_{\varepsilon\varepsilon} \end{pmatrix}.$$

The upper diagonal block need only be nonnegative definite, but it is assumed that the lower block is positive definite. The observable variables (y_i, x_i, Z_i) are assumed to have the positive definite covariance matrix

$$V = \begin{pmatrix} v_{yy} & v_{yx} & v_{yz} \\ v_{xy} & v_{xx} & v_{xz} \\ v_{zy} & v_{zx} & v_{zz} \end{pmatrix}.$$

Let L and U be numbers such that $-1 < L \leqslant \operatorname{corr}(u_i, \varepsilon_i) \leqslant U < 1$. The problem is to determine if a given vector (V, L, U) implies restrictions on θ and/or δ. It will be shown that some (V, L, U) do imply such restrictions, and formulas are given for the endpoints of the intervals characterizing these restrictions. The formulas depend on (V, L, U), and can be consistently estimated by replacing V with some consistent estimator $\hat{V}$.

The analysis relies on the fact that (1)–(2) imply equations expressing V as a function of θ, δ, and $\operatorname{var}(\chi_i, Z_i, u_i, \varepsilon_i)$. Using the identity $\sigma_{zz} \equiv v_{zz}$, and noting that the block diagonality of $\operatorname{var}(\chi_i, Z_i, u_i, \varepsilon_i)$ implies $\sigma_{z\chi} = v_{zx}$, these equations can be written

$$v_{yy} = \sigma_{\chi\chi}\theta^2 + \delta' v_{zz}\delta + 2\theta v_{xz}\delta + \sigma_{uu}, \tag{3}$$

$$v_{xx} = \sigma_{\chi\chi} + \sigma_{\varepsilon\varepsilon}, \tag{4}$$

$$v_{xy} = \sigma_{\chi\chi}\theta + v_{xz}\delta + \sigma_{u\varepsilon}, \tag{5}$$

$$v_{zy} = v_{zx}\theta + v_{zz}\delta. \tag{6}$$

Restrictions on $\rho \equiv \operatorname{corr}(u_i, \varepsilon_i)$ can be imposed on this equation system via the identities

$$(7) \qquad \rho^2 = \frac{\sigma_{u\varepsilon}^2}{\sigma_{uu}\sigma_{\varepsilon\varepsilon}},$$

$$(8) \qquad \operatorname{sign}(\rho) = \operatorname{sign}(\sigma_{u\varepsilon}).$$

Given (V, ρ), let A be the set of vectors $(\theta, \delta, \sigma_{\chi\chi}, \sigma_{uu}, \sigma_{\varepsilon\varepsilon}, \sigma_{ue})$ that solve (3)–(8) subject to $\operatorname{var}(\chi_i, Z_i)$ being n.n.d. and $\operatorname{var}(u_i, \varepsilon_i)$ being p.d. Let P be the projection of A on the θ-axis, and let Θ be the union of the sets P generated by letting ρ range over the interval $[L, U]$. The union Θ consists of all those, and only those, θ-values that are consistent with the given V, the prior restriction $L \leqslant \rho \leqslant U$, and the restrictions on $\operatorname{var}(\chi_i, Z_i, u_i, \varepsilon_i)$; in this sense Θ shall be regarded as the set of possible values for θ, and the complement to Θ, if nonempty, as representing restrictions on θ. Theorem 1 below characterizes Θ for all possible (V, L, U). Section 3 then maps Θ into restrictions on δ.

To state Theorem 1, let

$$r = \frac{v_{yx\cdot z}}{\sqrt{v_{yy\cdot z} v_{xx\cdot z}}}, \qquad b = \frac{v_{yx\cdot z}}{v_{xx\cdot z}}, \qquad b_r = \frac{v_{yy\cdot z}}{v_{yx\cdot z}},$$

where $v_{xx\cdot z} \equiv v_{xx} - v_{xz} v_{zz}^{-1} v_{zx}$, $v_{yy\cdot z} \equiv v_{yy} - v_{yz} v_{zz}^{-1} v_{zy}$, and $v_{yx\cdot z} \equiv v_{yx} - v_{yz} v_{zz}^{-1} v_{zx}$. The quantity r is the population partial correlation between y and x, while b and b_r are the population counterparts to, respectively, the coefficient on x from the least squares regression of y on (x, Z), and the reciprocal of the coefficient on y from the regression of x on (y, Z); recall that (b, b_r) is the Gini bound.

THEOREM 1: *Assume $r > 0$, and let*

$$m_U = \sqrt{bb_r}\left(R_U - \sqrt{R_U^2 - 1}\right), \qquad M_U = \sqrt{bb_r}\left(R_U + \sqrt{R_U^2 - 1}\right)$$

where $R_U = [r - \sqrt{(r^2 - U^2)(1 - U^2)}\,]/U^2$. Then:

(a) $U \leqslant 0 \Rightarrow \Theta = \{\theta\colon\ b < \Theta < b_r\}$;

(b) $0 < U < r \Rightarrow \Theta = \{\theta\colon\ m_U \leqslant \theta \leqslant M_U\}$ *and* $0 < m_U < b < b_r < M_U$;

(c) $L \leqslant r \leqslant U \Rightarrow \Theta = \{\theta\colon\ -\infty < \theta < \infty\}$;

(d) $L > r \Rightarrow \Theta = \{\theta\colon\ \theta < b$ *or* $\theta > b_r\}$.

REMARK 1: The assumption $r > 0$ is unrestrictive (assuming $r \neq 0$) since one can always multiply x_i by -1. Also, it can be shown that m_U and M_U are, respectively, decreasing and increasing in U, satisfying $(m_U, M_U) \to (b, b_r)$ as $U \to 0$, and $(m_U, M_U) \to (m_r, M_r)$ as $U \to r$, where

$$(9) \qquad m_r = b_r - \sqrt{b_r^2 - bb_r} \quad \text{and} \quad M_r = b_r + \sqrt{b_r^2 - bb_r}.$$

To use Theorem 1 for public reporting it is best to give readers a way of using their own values for (L, U). One such way, made possible by the fact that L does not appear in (a) and (b), is to report the bounds on θ as functions of U. Another way, appropriate when a hypothesis of the form $\theta > c$ (or $\theta < c$) is at issue, is to use the following result to report the set of all prior bounds that imply the hypothesis:

COROLLARY 1.1: *Let*

$$U_c = \sqrt{\frac{(rh - r^2 - 1)}{(h^2/4 - 1)}}, \qquad h = \frac{c}{\sqrt{bb_r}} + \frac{\sqrt{bb_r}}{c}.$$

(a) *If $c \leqslant m_r$, then $U < r \Rightarrow \theta > c$.*
(b) *If $m_r < c < b$, then $U < U_c \Rightarrow \theta > c$.*
(c) *If $c > b$, then there are no pairs (L, U) implying $\theta > c$.*
(d) *If $c \geqslant M_r$, then $U < r \Rightarrow \theta < c$.*
(e) *If $b_r < c < M_r$, then $U < U_c \Rightarrow \theta < c$.*
(f) *If $c < b_r$, then there are no pairs (L, U) implying $\theta < c$.*

REMARK 2: U_c is the value of U such that $m_U = c$ for c satisfying $m_r < c < b$; if instead $b_r < c < M_r$, then U_c is that U such that $M_U = c$. In either case, $0 < U_c < r$.

REMARK 3: Because $m_r > 0$, one can use part (a) for the important hypothesis $\theta > 0$.

Returning to Theorem 1, it is worth emphasizing part (d), which is potentially very useful. The Gini result is rarely invoked in practice because for many typical data sets it is too long to be useful. (This generalization does not apply to the intervals for δ; see Section 3.) For example, the interval is $(0.1, 10)$ if $v_{yx \cdot z} = 0.1$ and $v_{yy \cdot z} = v_{xx \cdot z} = 1$. Under part (d), however, such an interval is forbidden to θ, so its length can be an advantage in discriminating against many hypotheses.

3. RESTRICTIONS ON OTHER COEFFICIENTS

Often the reason a proxy is used is to avoid omitted variable bias in estimating coefficients on perfectly measured variables. Theorem 1 permits inferences about these coefficients via the following expression, which is obtained by solving (6) for δ:

$$\delta = v_{zz}^{-1} v_{zy} - v_{zz}^{-1} v_{zx} \theta. \tag{10}$$

Let δ_j, z_{ij}, q_j, and s_j denote the jth elements of, respectively, δ, Z_i, $v_{zz}^{-1} v_{zy}$, and $v_{zz}^{-1} v_{zx}$. Note that q_j is the coefficient on z_{ij} from the population least squares regression of y_i on Z_i, and s_j is the coefficient on z_{ij} from the regression of x_i on Z_i. The jth equation in (10) can now be written

$$\delta_j = q_j - s_j \theta.$$

The term $s_j \theta$ is the usual one for omitted variable bias. The value of a proxy is that it can limit the size of this bias. For example, if the conditions in (a) or (b) of Theorem 1 are satisfied, then $s_j \theta$, and hence δ_j, will be confined to a finite interval. In contrast to typical bounds on θ, it should not be unusual for bounds on δ_j to be quite short, because s_j can be near zero. Note also that bounds for δ_j need not exclude the origin, thus complicating inference about sign. (On the other hand, the "forbidden interval" for δ_j implied by (d) of Theorem 1 may include the origin, thereby allowing one to reject the oft-tested hypothesis $\delta_j = 0$.) The possibilities for sign inference, and more generally for inference about one-sided hypotheses, are given by the following summary of those pairs (L, U) that imply $\delta_j > c^*$ for given c^*:

COROLLARY 1.2: *Let $c = (q_j - c^*)/s_j$.*
(a) *If $c \leqslant m_r$ and $s_j < 0$, then $U < r \Rightarrow \delta_j > c^*$.*
(b) *If $m_r < c < b$ and $s_j < 0$, then $U < U_c \Rightarrow \delta_j > c^*$.*
(c) *If $c > b$ and $s_j < 0$, then there are no pairs (L, U) that can ensure $\delta_j > c^*$.*
(d) *If $c \geqslant M_r$ and $s_j > 0$, then $U < r \Rightarrow \delta_j > c^*$.*
(e) *If $b_r < c < M_r$ and $s_j > 0$, then $U < U_c \Rightarrow \delta_j > c^*$.*
(f) *If $c < b_r$ and $s_j > 0$, then there are no pairs (L, U) that can ensure $\delta_j > c^*$.*

4. CONCLUSION

Individuals must contemplate values for U and L to use these results. There is a representation for ρ^2 that eases this task in situations where u_i and ε_i are thought to be influenced by the same variables. Let W_i denote a vector of unobserved variables such that $u_i = \alpha W_i + v_i$ and $\varepsilon_i = \eta W_i + e_i$, where W_i, v_i, and e_i are mutually uncorrelated, and let R^2_{wu} and $R^2_{w\varepsilon}$ denote the associated population squared multiple correlation coefficients. It is straightforward to show that $\rho^2 = R^2_{wu} R^2_{w\varepsilon}$, implying that bounds on ρ^2 can be derived from bounds on R^2_{wu} and $R^2_{w\varepsilon}$. For example, consider the consumption function discussed in the introduction. One strategy is to simply set $U = \sqrt{B_{wu}}$, where B_{wu} is the upper bound on R^2_{wu}, i.e., it answers the question "what is the maximum proportion of transitory consumption variation I am willing to attribute to the variables in W_i?" If U is small enough to support an inference then no further assessment is needed. Otherwise, one can try $U = \sqrt{B_{wu} B_{w\varepsilon}}$, where $B_{w\varepsilon}$ is the answer to "what is the maximum proportion of transitory income variation I am willing to attribute to W_i?"

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APPENDIX

In what follows, inequalities are said to be "equivalent" if they have the same solution set. The lemmas used in part A are proven in part B.

A. PROOF OF THEOREM 1: Substitute (10) into (3)–(5) to eliminate δ, yielding

(11) $$v_{yy\cdot z} = \phi\theta^2 + \sigma_{uu},$$

(12) $$v_{xx\cdot z} = \phi + \sigma_{\varepsilon\varepsilon},$$

(13) $$v_{yx\cdot z} = \phi\theta + \sigma_{u\varepsilon},$$

where

$$\phi \equiv \sigma_{\chi\chi} - v_{xz} v_{zz}^{-1} v_{zx}.$$

P can now be found as the projection of the set of vectors $(\theta, \phi, \sigma_{uu}, \sigma_{\varepsilon\varepsilon}, \sigma_{u\varepsilon})$ that solve (7), (8), and (11)–(13) subject to $\operatorname{var}(\chi_i, Z_i)$ being n.n.d. and $\operatorname{var}(u_i, \varepsilon_i)$ being p.d. When $\rho^2 > 0$ the only binding constraint these latter requirements place on the solution is

(14) $$\phi \geqslant 0,$$

which ensures that $\operatorname{var}(\chi_i, Z_i)$ is n.n.d.; the positive definiteness of $\operatorname{var}(u_i, \varepsilon_i)$ is always satisfied by the solutions, a necessary consequence of evaluating (7) with $0 < \rho^2 < 1$, and the assumption that V is p.d. When $\rho = 0$ the inequalities $\sigma_{uu} > 0$ and $\sigma_{\varepsilon\varepsilon} > 0$ are also binding constraints.

Next use (11)–(13) to eliminate σ_{uu}, $\sigma_{\varepsilon\varepsilon}$, and $\sigma_{u\varepsilon}$ from (8)–(7), yielding

(15) $$\operatorname{sign}(\rho) = \operatorname{sign}(v_{yx\cdot z} - \phi\theta)$$

and $\rho^2 = (v_{yx\cdot z} - \phi\theta)^2 / [(v_{yy\cdot z} - \phi\theta^2)(v_{xx\cdot z} - \phi)]$. Multiplying this last equation by its right hand side denominator and rearranging yields

(16) $$A\phi^2 + B\phi + C = 0, \quad \text{where}$$

$$A = (1-\rho^2)\theta^2, \qquad B = \rho^2 v_{xx\cdot z}\theta^2 - 2v_{yx\cdot z}\theta + \rho^2 v_{yy\cdot z}, \qquad C = (r^2 - \rho^2) v_{yy\cdot z} v_{xx\cdot z}.$$

For $\rho^2 > 0$ the set P can now be found as the projection of the set of vectors (θ, ϕ) that satisfy (14)–(16). For $\rho = 0$, P is the projection of the solutions to (14), (16), and $(v_{yy\cdot z} - \phi\theta^2) > 0$ and

$(v_{xx\cdot z}-\phi)>0$.

The remainder of the proof is in four parts. Part I determines the solution set to (16). This set takes four forms, depending on whether ρ satisfies $\rho=0$, $0<\rho^2<r^2$, $\rho^2=r^2$, or $r^2<\rho^2<1$. Part II finds the subset of this solution set that satisfies (14), and, if $\rho=0$, $(v_{yy\cdot z}-\phi\theta^2)>0$ and $(v_{xx\cdot z}-\phi)>0$ as well. Part III finds the subset that satisfies (15) also, and then obtains P as the projection of this subset. Part IV obtains Θ as the union of the sets P implied by each ρ in $[L,U]$.

I. For any given $\theta\neq 0$ the coefficient A in (16) is positive and the equation is a quadratic in ϕ with solutions

$$\bar{\phi}_{(\theta)}=\frac{-B+\sqrt{B^2-4AC}}{2A}, \tag{17}$$

$$\underline{\phi}_{(\theta)}=\frac{-B-\sqrt{B^2-4AC}}{2A}. \tag{18}$$

These are defined if and only if $B^2-4AC\geqslant 0$; to determine those values of θ for which this holds substitute from (16) to obtain

$$B^2-4AC=\left(\rho^2 v_{xx\cdot z}\theta^2-2v_{yx\cdot z}\theta+\rho^2 v_{yy\cdot z}\right)^2-4(1-\rho^2)\theta^2(r^2-\rho^2)v_{yy\cdot z}v_{xx\cdot z}. \tag{19}$$

Inspecting the second term shows that if ρ satisfies $r^2<\rho^2<1$ then $B^2-4AC>0$ for all θ. For the case $\rho^2=r^2$ the second term of (19) vanishes, so that $B^2-4AC=B^2\geqslant 0$. (The equality $B^2=0$ holds if and only if θ equals one of the two roots of B; in general these roots are given by

$$w_1=\sqrt{\frac{v_{yy\cdot z}}{v_{xx\cdot z}}}\left(\frac{r}{\rho^2}-\sqrt{\left(\frac{r}{\rho^2}\right)^2-1}\right),\qquad w_2=\sqrt{\frac{v_{yy\cdot z}}{v_{xx\cdot z}}}\left(\frac{r}{\rho^2}+\sqrt{\left(\frac{r}{\rho^2}\right)^2-1}\right),$$

and if $\rho^2=r^2$ then $w_1=m_r$ and $w_2=M_r$, where (9) gives m_r and M_r.) Thus $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ are defined for all nonzero θ when $\rho^2>r^2$ or $\rho^2=r^2$. To analyze the case $0<\rho^2<r^2$ we need the following results:

Lemma 1.1: *If* $0<\rho^2<r^2$ *and* $r>0$ *then* $B^2-4AC=0$ *has four real roots, given by*

$$m=\sqrt{bb_r}\left(R-\sqrt{R^2-1}\right), \tag{20}$$

$$M=\sqrt{bb_r}\left(R+\sqrt{R^2-1}\right), \tag{21}$$

$$z_1=\sqrt{bb_r}\left(W-\sqrt{W^2-1}\right),$$

$$z_2=\sqrt{bb_r}\left(W+\sqrt{W^2-1}\right),$$

where

$$R=\frac{r-\sqrt{(r^2-\rho^2)(1-\rho^2)}}{\rho^2}, \tag{22}$$

$$W=\frac{r+\sqrt{(r^2-\rho^2)(1-\rho^2)}}{\rho^2}.$$

Lemma 1.2: *If the assumptions of Lemma* 1.1 *hold, then*: (a) $0<z_1<m<M<z_2$; (b) $B^2-4AC\geqslant 0$ *if and only if* $\theta\leqslant z_1$ *or* $m\leqslant\theta\leqslant M$ *or* $\theta\geqslant z_2$.

Thus, when $0<\rho^2<r^2$ the functions $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ are defined for nonzero θ if and only if $\theta\leqslant z_1$ or $m\leqslant\theta\leqslant M$ or $\theta\geqslant z_2$. Finally, note that $\rho=0$ implies $B^2-4AC=0$ for all θ, so that $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ reduce to $-B/2A=v_{yx\cdot z}/\theta$, which is defined for all nonzero θ.

If $\theta = 0$, then $A = 0$, so (16) is linear in ϕ, and can be solved, except when $\rho = 0$:

$$\phi_{(0)} = v_{xx\cdot z}\left(1 - \frac{r^2}{\rho^2}\right). \tag{23}$$

II. The case $r^2 < \rho^2 < 1$: Inspecting the second term of (19) reveals that $r^2 < \rho^2 < 1$ implies $\sqrt{B^2 - 4AC} > |B|$, which together with the fact that $A > 0$ for all nonzero θ, implies via (17) and (18) that $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ are, respectively, positive and negative for all nonzero θ. Also, note from (23) that $\theta = 0$ implies $\phi_{(0)} > 0$ in this case.

The case $\rho^2 = r^2$: Expression (19) implies $\sqrt{B^2 - 4AC} = |B|$, which together with the fact that B is negative on (m_r, M_r) implies that for nonzero θ-values $\bar{\phi}_{(\theta)}$ equals zero except on the interval (m_r, M_r), over which it is positive, and that $\underline{\phi}_{(\theta)}$ is negative except on the interval $[m_r, M_r]$, over which it equals zero. For $\theta = 0$, expression (23) gives $\phi_{(0)} = 0$.

The case $0 < \rho^2 < r^2$: Lemma 1.2 says $\sqrt{B^2 - 4AC}$ is defined only for a subset of nonzero θ-values. For such values (19) implies $\sqrt{B^2 - 4AC} < |B|$, establishing the following: if $B < 0$ then $\underline{\phi}_{(\theta)} > 0$ and $\bar{\phi}_{(\theta)} > 0$, whereas if $B > 0$ then $\underline{\phi}_{(\theta)} < 0$ and $\bar{\phi}_{(\theta)} < 0$. Recall that $B < 0$ if and only if $w_1 < \theta < w_2$, and note the following:

LEMMA 1.3: *The conditions of Lemma* 1.1 *imply* $z_1 < w_1 < m < b < b_r < M < w_2 < z_2$.

It follows from this and Lemma 1.2 that $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ are real and nonnegative if and only if $m \leqslant \theta \leqslant M$. For $\theta = 0$, expression (23) gives $\phi_{(0)} < 0$.

The case $\rho = 0$: Recall that $\theta = 0$ cannot be part of a solution to (16). For $\theta \neq 0$ recall that $\rho = 0 \Rightarrow \bar{\phi}_{(\theta)} = \underline{\phi}_{(\theta)} = v_{yx\cdot z}/\theta$, which is nonnegative if and only if $\theta > 0$. For this case we must also impose the constraints $(v_{yy\cdot z} - \phi\theta^2) > 0$ and $(v_{xx\cdot z}\phi) > 0$. Substituting $\phi = v_{yx\cdot z}/\theta$ into the latter inequality and rearranging yields $\theta > v_{yx\cdot z}/v_{xx\cdot z} \equiv b$. Substituting $\phi = v_{yx\cdot z}/\theta$ into the former inequality gives $\theta < v_{yy\cdot z}/v_{yx\cdot z} \equiv b_r$. These inequalities on θ directly give the projection P, which is the Gini interval. The parent solution set is depicted in Figures 1–3 as that part of the hyperbola lying beneath $\bar{\phi}_{(\theta)}$.

III. The final step in obtaining P is to impose (15). The results are stated first; when reading the proofs that follow, it may be useful to refer to Figures 1–3.

(i) If $\rho \leqslant 0$, then $P = (b, b_r)$.
(ii) If $0 < \rho < r$, then $P = [m, M]$.
(iii) If $\rho = r$, then $P = (-\infty, \infty)$.
(iv) If $r < \rho < 1$, then $P = \{\theta\colon \theta < b \text{ or } \theta > b_r\}$.

Proof of (i): That $\rho = 0 \Rightarrow P = (b, b_r)$ is the Gini result. To establish $\rho < 0 \Rightarrow P = (b, b_r)$, note from Lemma 1.3 that $m < b < b_r < M$ holds when $0 < \rho^2 < r^2$. Together with the other results of Parts I–II, this implies that for any ρ^2 the function $\bar{\phi}_{(\theta)}$ is defined and satisfies $\phi \geqslant 0$ on the interval (b, b_r). It thus suffices to show $\bar{\phi}_{(\theta)}$ satisfies $v_{yx\cdot z} - \phi\theta < 0$ if and only if $\theta \in (b, b_r)$, and that $\underline{\phi}_{(\theta)}$ does not satisfy $v_{yx\cdot z} - \phi\theta < 0$ for any θ. To show the former first substitute the right hand side of (17) into $v_{yx\cdot z} - \phi\theta$ to obtain

$$v_{yx\cdot z} - \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right)\theta, \tag{24}$$

which after using (16) to eliminate A from the denominator, and then rearranging, equals

$$\frac{2v_{yx\cdot z}(1 - \rho^2)\theta + B - \sqrt{B^2 - 4AC}}{2(1 - \rho^2)\theta}.$$

The denominator has the same sign as θ, so we must show that when $\theta > 0$ the numerator is negative if and only if $b < \theta < b_r$, and that the numerator is negative for all $\theta < 0$. Clearly, the numerator is negative if and only if

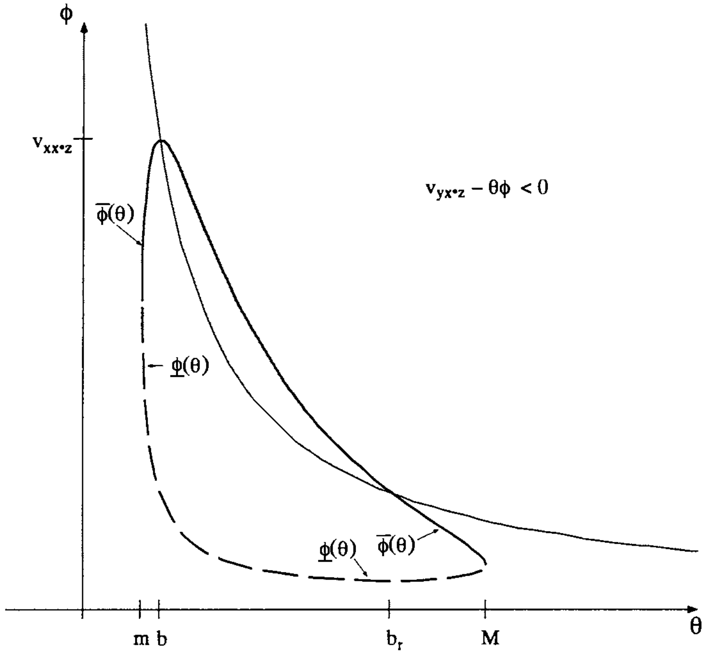

FIGURE 1.—The case $0 < \rho^2 < r^2$. The matrix V used to generate this figure is also used for Figures 2–3.

$$2v_{yx\cdot z}(1-\rho^2)\theta + B < \sqrt{B^2 - 4AC}.$$

Using (16) to eliminate B from the left hand side gives the equivalent inequality

$$\rho^2\left(v_{xx\cdot z}\theta^2 - 2v_{yx\cdot z}\theta + v_{yy\cdot z}\right) < \sqrt{B^2-4AC}. \tag{25}$$

The parenthesis on the left is positive for all θ, so this inequality holds if and only if

$$\rho^4\left(v_{xx\cdot z}\theta^2 - 2v_{yx\cdot z}\theta + v_{yy\cdot z}\right)^2 < B^2 - 4AC.$$

Substituting (19) for $B^2 - 4AC$ on the right hand side and then rearranging yields

$$4v_{xx\cdot z}v_{yx\cdot z}\rho^2(1-\rho^2)\theta(\theta - b)(\theta - b_r) < 0. \tag{26}$$

By assumption $\rho \neq 0$ and $v_{yx\cdot z} > 0$, so this holds for all negative values of θ, and holds for positive values if and only if $b < \theta < b_r$.

It remains to show that $\underline{\phi}_{(\theta)}$ cannot satisfy $v_{yx\cdot z} - \phi\theta < 0$ for any θ. Substitute the right hand side of (18) into $v_{yx\cdot z} - \phi\theta$, and then follow the same steps as between (24)–(25) to obtain the following inequality, which is equivalent to $v_{yx\cdot z} - \underline{\phi}_{(\theta)}\theta < 0$:

$$\rho^2\left(v_{xx\cdot z}\theta^2 - 2v_{yx\cdot z}\theta + v_{yy\cdot z}\right) < -\sqrt{B^2-4AC}.$$

But this is impossible, since the left hand side is positive for all θ; hence $v_{yx\cdot z} - \underline{\phi}_{(\theta)}\theta < 0$ is also impossible.

Proof of (ii): Recall that if $0 < \rho^2 < r^2$ then $\bar{\phi}_{(\theta)}$ and $\underline{\phi}_{(\theta)}$ are defined, and satisfy $\phi \geqslant 0$, if and only if $m \leqslant \theta \leqslant M$. It thus suffices to show that $\underline{\phi}_{(\theta)}$ satisfies $v_{yx\cdot z} - \phi\theta > 0$ if $m \leqslant \theta \leqslant M$. Substituting the right side of (18) into $v_{yx\cdot z} - \phi\theta$ and then rearranging as was done from (24) to (25), except using the reverse inequality sign, yields an inequality which is true if and

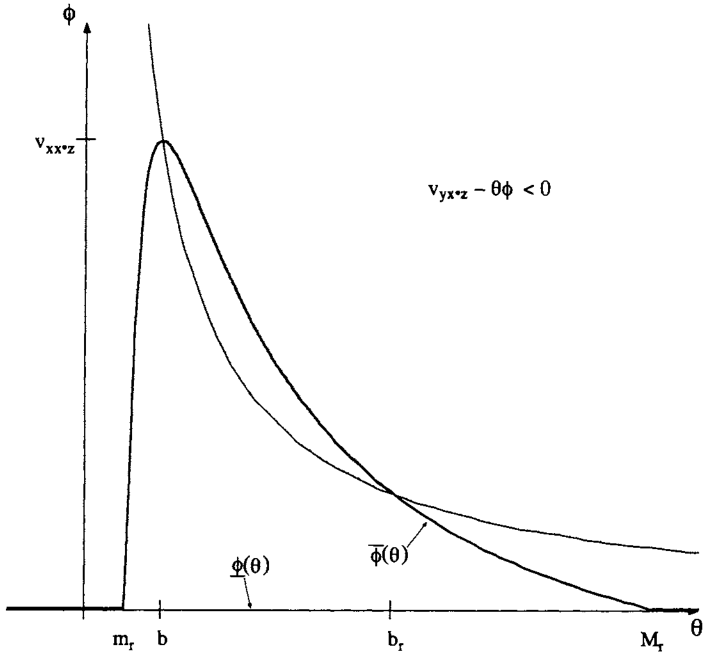

FIGURE 2.—The case $\rho^2 = r^2$. Here $\underline{\phi}_{(\theta)}$ coincides with the horizontal axis for $m_r \leqslant \theta \leqslant M_r$; elsewhere it is negative. The function $\overline{\phi}_{(\theta)}$ coincides with the horizontal axis for $\theta < m_r$ and $\theta > M_r$.

only if $v_{yx \cdot z} - \underline{\phi}_{(\theta)}\theta > 0$ is true:

$$\rho^2\left(v_{xx \cdot z}\theta^2 - 2v_{yx \cdot z}\theta + v_{yy \cdot z}\right) > -\sqrt{B^2 - 4AC}\,.$$

Recalling that the left side is positive for all θ establishes that this inequality is true everywhere the right hand side is defined, which is the interval $[m, M]$.

Proof of (iii): Recall that $\rho^2 = r^2$ implies $\overline{\phi}_{(\theta)} = 0$ for all nonzero θ outside the interval (m_r, M_r), $\underline{\phi}_{(\theta)} = 0$ everywhere in $[m_r, M_r]$, and $\phi_{(0)} = 0$. Thus, $v_{yx \cdot z} - \phi\theta > 0$ and $\phi \geqslant 0$ are satisfied by every θ.

Proof of (iv): The solution $(0, \phi_{(0)})$ satisfies both $\phi \geqslant 0$ and $v_{yx \cdot z} - \phi\theta > 0$. For nonzero θ, recall that $r < \rho < 1$ implies $\underline{\phi}_{(\theta)} < 0$ and $\overline{\phi}_{(\theta)} > 0$ for all $\theta \neq 0$. It thus suffices to show that $\overline{\phi}_{(\theta)}$ satisfies $v_{yx \cdot z} - \phi\theta > 0$ if and only if $\theta < b$ or $\theta > b_r$. Recall from the proof to (i) that $v_{yx \cdot z} - \theta\overline{\phi}_{(\theta)}$ equals (24), which for $\theta > 0$ has the same sign as the left hand side of (26), which is positive if and only if $0 < \theta < b$ or $\theta > b_r$. If $\theta < 0$ then $v_{yx \cdot z} - \theta\overline{\phi}_{(\theta)}$ has the sign opposite to that of the left hand side of (26), which is negative for all $\theta < 0$.

IV. The set Θ is the union of the sets P implied by every ρ in the interval $[L, U]$. Part (a) of Theorem 1 thus follows because P is invariant to nonpositive ρ, as reported in (i) above. Part (d) follows in the same way from (iv), and (c) is implied by (iii). To prove (b) recall that $0 < m < b < b_r < M$; hence, (i) and (ii) imply Θ equals the union of the P corresponding to ρ in $(0, U]$. Because the pair (m_U, M_U) equals (m, M) evaluated at $\rho^2 = U^2$, it suffices to show that m and M are, respectively, strictly decreasing and strictly increasing functions of ρ^2 on $(0, r^2)$. This is done by showing that $dm/d\rho^2 < 0$ and $dM/d\rho^2 > 0$ for all $\rho^2 \in (0, r^2)$. Note from (21) that $dM/dR > 0$, and refer to the proof of Lemma 1.2 to see that $dm/dR < 0$. It thus suffices to show $dR/d\rho^2 > 0$; this is done by signing the derivative of (22), a demonstration available on request. *Q.E.D.*

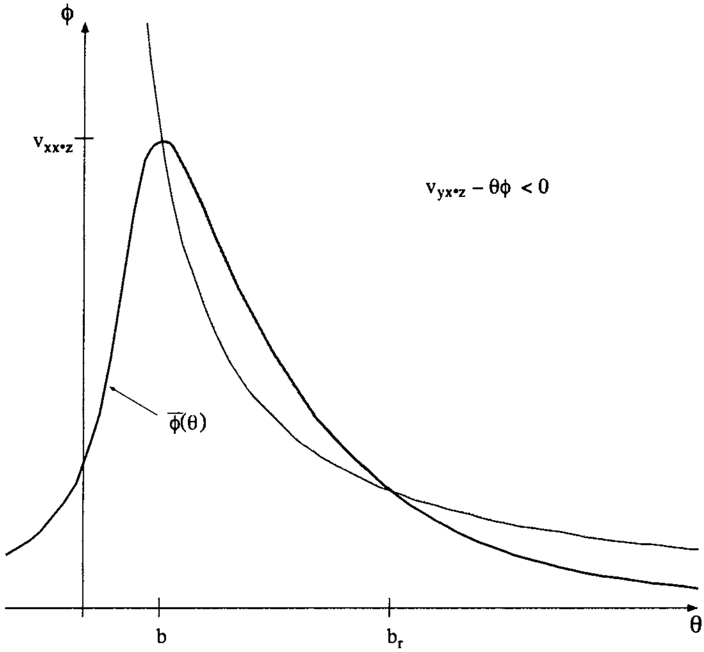

FIGURE 3.—The case $r^2 < \rho^2 < 1$. The function $\underline{\phi}_{(\theta)}$ is not depicted, as it is negative for all θ.

B. PROOF OF LEMMA 1.1: Rearranging the right side of (19) gives the polynomial

$$v_{xx\cdot z}^2 \rho^4 \theta^4 - 4 v_{xx\cdot z} v_{yx\cdot z} \rho^2 \theta^3$$

$$+ v_{yy\cdot z} v_{xx\cdot z} \left(4r^2\rho^2 + 4\rho^2 - 2\rho^4\right)\theta^2 - 4 v_{yy\cdot z} v_{yx\cdot z} \rho^2 \theta + v_{yy\cdot z}^2 \rho^4.$$

The reader can confirm that if $0 < \rho^2 < r^2$ then this polynomial equals $\rho^4 Q_1 Q_2$, where

$$Q_1 = v_{xx\cdot z}\theta^2 - 2\sqrt{v_{yy\cdot z} v_{xx\cdot z}}\, W\theta + v_{yy\cdot z},$$

$$Q_2 = v_{xx\cdot z}\theta^2 - 2\sqrt{v_{yy\cdot z} v_{xx\cdot z}}\, R\theta + v_{yy\cdot z}.$$

The quadratic Q_1 has roots (z_1, z_2), and Q_2 has roots (m, M). For these to be real it suffices that $W^2 > 1$ and $R^2 > 1$. By inspection $W > R$, so it suffices to show $R > 1$. Using (22) this can be written as $[r - \sqrt{(r^2 - \rho^2)(1 - \rho^2)}\,]/\rho^2 > 1$, which can be shown to be equivalent to $(r-1)^2 > 0$, which holds because positive definite V implies $r < 1$. *Q.E.D.*

PROOF OF LEMMA 1.2: By inspection z_1, z_2, m, and M are all positive, and $M < z_2$. (Recall that $W > R > 1$.) It is obvious that $m < M$. Next note that $dm/dR = [(\sqrt{R^2 - 1} - R)/\sqrt{R^2 - 1}\,] < 0$. Because $W > R$, it follows that $z_1 < m$. Now recall that $B^2 - 4AC = \rho^4 Q_1 Q_2$, where Q_1 has roots (z_1, z_2) and Q_2 has (m, M). The coefficients on θ^2 in Q_1 and Q_2 are both positive, so $Q_1 < 0$ iff $z_1 < \theta < z_2$, and $Q_2 < 0$ iff $m < \theta < M$. Together with the ordering of z_1, z_2, m, and M established above, this implies that $B^2 - 4AC \geqslant 0$ iff $\theta \leqslant z_1$ or $m \leqslant \theta \leqslant M$ or $\theta \geqslant z_2$. *Q.E.D.*

PROOF OF LEMMA 1.3: The quantity w_1 differs from m only in that r/ρ^2 replaces R. Since $R < r/\rho^2 < W$, the argument used above to prove $z_1 < m$ also establishes $z_1 < w_1 < m$. The

inequalities $M < w_2 < z_2$ and $b < b_r$ are obvious. The inequality $m < b$ can be established by showing its equivalence to an obviously true inequality. Multiply $m < b$ by $\sqrt{v_{xx\cdot z}/v_{yy\cdot z}}$ and rearrange into the equivalent inequality $R - r < \sqrt{R^2 - 1}$. Since $R - r$ is positive, squaring both sides gives the equivalent inequality, $R^2 - 2rR + r^2 < R^2 - 1$. Subtracting $R^2 + r^2$ yields $-2rR < -r^2 - 1$. Using (22) to eliminate R, and adding $2r^2/\rho^2$ to both sides yields $2r\sqrt{(r^2 - \rho^2)(1 - \rho^2)}/\rho^2 < 2r^2/\rho^2 - r^2 - 1$. Multiplying by ρ^2 gives

$$2r\sqrt{(r^2 - \rho^2)(1 - \rho^2)} < 2r^2 - r^2\rho^2 - \rho^2. \tag{27}$$

The right side is positive since $\rho^2 < r^2$; hence squaring both sides gives another equivalent inequality; doing so, and then rearranging, yields $0 < r^4 - 2r^2 + 1$, which is equivalent to $0 < (r^2 - 1)^2$. To establish $b_r < M$ in a similar fashion, multiply both sides by $\sqrt{v_{xx\cdot z}/v_{yy\cdot z}}$ to obtain $1/r < R + \sqrt{R^2 - 1}$. If $R \geqslant 1/r$ this inequality is obviously true. To show it is true when $R < 1/r$, subtract R from both sides to obtain $1/r - R < \sqrt{R^2 - 1}$; by assumption the left side is positive, so squaring both sides yields the equivalent inequality $1/r^2 - 2R/r + R^2 < R^2 - 1$. Adding $2/\rho^2 - R^2 - 1/r^2$ and using (22) to eliminate R yields $2\sqrt{(r^2 - \rho^2)(1 - \rho^2)}/r\rho^2 < 2/\rho^2 - 1/r^2 - 1$. Finally, multiplying by $r^2\rho^2$ yields the equivalent inequality (27), which was shown to be true. *Q.E.D.*

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